UFR-HEP 00/21
LMU-TPW 24/00

Fractional Quantum Hall Excitations as RdTS Highest Weight State Representations

I. Benkaddour [1,2], A. ELRhalami [1] and E.H.Saidi[1]
[1] *High Energy Physics Laboratory and UFR, Faculty of Sciences, Av Ibn Batota Po.Box 1014, Rabat, Morocco.*
[2] *Sektion der Physik Theorie, Muenchen Universitat, Muenchen, Thereseinstr 37, D-80333, Muenchen,Deutschland.*

e-mail: ilham@theorie.physik.uni-muenchen.de
arhalami@usa.net
H-saidi@fsr.ac.ma

(February 1, 2008)

Using the Chern-Simons effective model of fractional quantum Hall (FQH) systems, we complete partial results obtained in the literature on FQHE concerning topological orders of FQH states. We show that there exists a class of effective FQH models having the same filling fraction ν, interchanged under $Gl(n,Z)$ transformations and extends results on Haldane hierarchy. We also show that Haldane states at any generic hierarhical level n may be realised in terms of n Laughlin states composites and rederive results for the n=2,3 levels respectively associated with $\nu=\frac{2}{5}$ and $\nu=\frac{3}{7}$ filling fractions. We study symmetries of the filling fractions series $\nu=\frac{p_2}{p_1p_2-1}$ and $\nu=\frac{p_1p_2-1}{p_1p_2p_3-p_1-p_2}$, with p_1 odd and p_2 and p_3 even integers, and show that, upon imposing the Gl(n,Z) invariance, we get remarkable informations on their stability. Then, we reconsider the Rausch de Traubenberg and Slupinsky (RdTS) algebra recently obtained in [1,2] and analyse its limit on the boundary $\partial(AdS_3)$ of the (1+2) dimensional manifold AdS_3. We show that generally one may distinguish bulk highest weight states (BHWS) living in AdS_3 and edge highest weight states (EHWS)living on the border $\partial(AdS_3)$. We explore these two kinds of RdTS representations carrying fractional values of the spin and propose them as condidates to describe the FQH states.

I. INTRODUCTION

Recently a non trivial generalisation of the $(1+2)$ dimensional Poincaré algebra going beyond the standard supersymmetric extension has been obtained in [1,2]. In addition to the usual (1+2)d translation $P_{0,\pm}$ and rotation $J_{0,\pm}$ symmetry generators, this extension refered herebelow to as the Slupinski-Traubenberg algebra (RdTS algebra for short), involves other kinds of conserved charges Q_s and $\bar{Q}_s$ transforming as $so(1,2)$ Verma modules of spin $s=\pm\frac{1}{k}; k\geq 2$. RdTS fractional supersymmetry (fsusy) is a new algebraic structure extending the standard structures in two dimensions [3,4,5,6,7]. It involves a non abelian $SO(1,2)$ rotation goup, including the $SO(2)$ abelian subgroup appearing in the usual constructions, and it is suspected to carry precious informations on quantum physics in (1+2)dimensional systems with boundaries; in particular in:
(**a**) classifying and characterising the quasiparticles used in hierarchy building of models of fractional quantum Hall effect (FQHE).
(**b**) the study the propagation of strings on $AdS_3\times N^{d-3}$; the three dimensional anti de Sitter background times a compact (d-3)-dimensinal manifold where d=26 for bosonic string and d=10 for type II superstrings.

Motivated by the following two:
(α) similarities observed in [8] between RdTS highest weight representations and quasiparticle states one encounters in models of fractional quantum Hall liquids,
(β) exist no consistent quantum field theoretical model yet of FQH liquids leading naturally to FQH quasiparticle states; that is FQH excitations carrying fractional values of the spin that are generated by an (q-deformed) algebra of creation and annihilation operators of certain quantum fields,
we want to explore in this paper the issue of interpreting the FQH excitations as RdTS highest weight states and the RdTS algebra as the algebra of creation and annihilation operators of an hypothetic quantum field model. In the present study we will also be interested in analysing somes aspects on FQH hierarchies. In what follows we propose to explicit a little bit our main motivations and the purposes of our study :

A. Main Motivations

In[8], we have studied aspects of the point (**b**) concerning the origin of RdTS symmetry and its link with space time boundary conformal invariance on ∂AdS_3. Here we

want to analyse point (**a**) but also complete partial results obtained in the literature on FQHE regarding topological orders of FQH states. For these topological orders, one should note that they are generally classified by three rational numbers namely the filling fraction ν, the electric charge Q_e and the spin $s = \frac{\theta}{\pi}$ or equivalently by the hierarchical matrix $\mathbf{K}$, the charge vector $\mathbf{t}$ and the spin vector $\mathbf{S}$. As far as the filling fraction ν is concerned, we want to show that there exists in fact a class of effective FQHE models having the same ν but not necessary the same kind of fundamental quasiparticles (elementary excitations). These models are interchanged by $Gl(n, Z)$ transformations and turn out to englobe results on the leading orders of the Haldane hierarchy realised in terms of composites of Laughlin states. The $Gl(n, Z)$ symmetry appearing here is unusual, goes beyond the $Sl(n, Z)$ symmetry of the $\mathbf{Z^n}$ charge lattice and carries remarkable informations on the stability of the filling fraction states. Before going into details, let us give other motivations supporting our interest in point (**a**). The basic idea is that in effective models of FQHE and RdTS algebra representations one has various quantities with striking analogies. In the Chern Simons (CS) effective model of FQHE, one encounters objects such as bulk states and edge states carrying fractional values of the spin as well as limits such the FQH droplet approximation allowing to make several estimations by using the conformal field theory (CFT) living on the border of the droplet. In the study of the RdTS algebra, one also has similar quantum states with the good features and, like for the droplet approximation of FQH liquids, one may here also make an analogous approximation by considering an appropriate (1+2)dimensional space time manifold M with a boundary conformal invariance on ∂M. To have a more insight on this correspondence between FQH Chern-Simons effective field model and RdTS operators algebra, let us anticipate on some of our results by giving the following two:

(**i**) Standard representations of the RdTS algebra in (1+2) dimensions have quantum states carrying fractional values of the spin. However, for (1+2) dimensional space time manifolds M with a non zero boundary ∂M and under a assumption to be specified later, the Q_s and $\bar{Q}_s$ charge operators may have different interpretations. They can be interpreted either as, $\mathbf{Q_s^+}$ and $\mathbf{Q_s^-}$, generators of (1+2) dimensional quantum field states carrying fractional values of the spin or as $\mathbf{q_s^+}$ and $\mathbf{q_s^-}$ and associated with highest weight states of a 2d boundary invariance. We shall refer to the first kind of states generated by $\mathbf{Q_s^+}$ and $\mathbf{Q_s^-}$ as bulk highest weight states (BHWS for short) while the second ones generated by $\mathbf{q_s^+}$ and $\mathbf{q_s^-}$ will be called edge highest weight states (EHWS). They live respectively in M and ∂M.

(**ii**) In the CS effective model of FQHE, one also distinguish two kinds of states carrying fractional values of the spin: Bulk states and edge ones. The first ones are localised states; they are used in the (1+2)dimensional effective CS gauge theory in the study of hierarchies. The second states are extended states and are described by a boundary conformal field theory in the droplet approximation. Edge states, which may also condensate, are responsible for the quantization of the Hall conductivity σ_{xy} as well as for the dynamics of the excitations on the boundary.

From this presentation, one sees that bulk (edge) excitations of the CS effective model of FQHE share several commun features with bulk (edge) representations of the RdTS algebra. They carry fractional spins and, up to an appropriate choice of the underlying geometry, are related to boundary conformal invariances living on the border of this geometry. One also learns that it may be possible to establish a correspondence between the CS effective field theory of FQH droplets and RdTS algebraic operators on AdS_3.

B. Purpose and Presentation of the Work

The purpose of the present paper is to answer some of the questions rised above. It uses both quantum field theoretical methods and algebraic constructions and aims to reach two objectives:

(α) First we want to complete partial results obtained in the literature on FQHE concerning topological orders of FQH states. More precisely we want to study aspects of topological properties of a generic Haldane state[9] at hierarhical level n in terms of orders of composites of n Laughlin states. Relaxing the $SL(n, Z)$ symmetry of the Z^n lattice of the charges of the CS $U(1)^n$ effective gauge theory of FQHE by allowing $GL(n, Z)$ transformations, we show by explicit computations that states of filling fraction $\nu_H = K_{11}^{-1}$ in the level n of Haldane hierarchy are realised in terms of composites of n Laughlin states of filling fractions ν_j; $j = 1, \dots, n$. We derive the general expression of the ν_j series and study their symmetries. We also give details for the two leading series $\nu = \frac{p_2}{p_1 p_2 - 1}$ and $\nu = \frac{p_1 p_2 - 1}{p_1 p_2 p_3 - p_1 - p_2}$, with p_1 odd and p_2 and p_3 even integers, and use the $Gl(n, Z)$ symmetries to derive informations on the stability of the filling fraction states.

(β) Second we examine the connection between RdTS representation states and those involved the CS effective theory of FQHE which we have described in point(ii) given subsection (1.1). Too particularly, we show that quasiparticles of fractional spins involved in the building of FQHE hierarchies could be interpreted as highest weight representation (HWR) states of the RdTS invariance. We also give arguments supporting a possible correpondence between FQH droplets and RdTS on AdS_3.

The presentation of this paper is as follows: In section 2, we review some general results of (1+2) dimensional effective CS model of the fractional quantum Hall liquids useful for our present study. We give the essential about FQHE features one needs and recall the relevant properties of fractional quantum Hall states as well as

topological orders and hierarchies. In subsection 2.2, we use the droplet approximation and we study the edge excitations of FQH droplets in terms of vertex operators living on the boundary of the droplet. In section 3, we reexamine the Haldane hierarchy of FQH droplets by taking into account the dynamics of the edge excitations as well as the interactions of branches. We first use appropriate $GL(n, Z)$ changes of CS gauge variables, going beyond the $SL(n, Z)$ symmety of the hypercubic Z^n charge lattice, to reinterpret a generic level n Haldane state of filling fraction ν_H as a composite of n Laughlin states of filling fractions ν_i; $i = 1, \ldots, n$. We show amongst others that $\nu_H = \sum_{i=0}^{n-1} \frac{1}{m_i m_{i+1}}$, where the m_i integers are the solutions of the series $m_i = p_i m_{i-1} - m_{i-2}$; with $m_0 = 1$ and $m_1 = p_1$ and have the property that the $m_i m_{i+1}$ product is usually an odd integer. We also discuss the effect of $Gl(n, Z)$ transformations on the Haldane series and the stability of the states. Then, we consider the branch interactions and show that the full hamiltonian is diagonalised by introducing effective velocities. In sections 4 and 5, we review the main lines of the construction of the RdTS algebra living in a (1+2) dimensional manifold with a boundary and show in section 6 why its HWR states may be viewed as condidates to describe bulk and edge excitations of the Chern Simons effective field theory of FQH liquids. In section 7, we make a discussion and give our conclusion.

II. GENERALITIES ON FQH LIQUIDS.

Roughly speaking, quantum Hall systems are defined as systems of electrons confined in a two dimensional layer embeded in a perpendicular external magnetic field B taken sufficiently strong so that electrons living on are totally polarized. The fractional Hall system should be incompressible [10], a feature which physically is interpreted as due to the existence of a positive energy gap for certain critical values of the filling fraction ν. The latter is given by the number of electron N_e divided by the number N_{ϕ_0} of unit of quantum flux ϕ_0. For each critical value ν, we have to distinguish localized and non localised states [13]. Localised states are bulk states described in the field approach by a (1+2)dimensional effective CS abelian gauge theory while non localised states are extended edge states described, in the droplet model, by a two dimensional boundary quantum field theory. Extended edge states are the carriers of the electric charge responsible of the non zero Hall conductivity σ_{xy} and its quantization [14]. The Hall current I_h which is proportional to the filling fraction ν comes from the non localized states and is believed to arise from skipping orbits of cyclotronic electrons elastically scattered by the edge potential barrier [10]. Non localized states are generated by both bulk and edge elementary exitations carrying fractional spins $s = \frac{\theta}{\pi}$ and fractional electric charges Q[11,15,16,17,18]. Non localized states condensate for some critical values of ν and it is admited that the Hall plateau of crtical value ν observed in experiments is indeed associated with the condensation of excitations. In FQH hierarchical models, we may have successive condensations of quasiparticles as in Haldane,Halperin, Jain and Zee hierarchies. We shall consider hereafter the Haldane hierarchy based on the Laughlin ground state of filling fraction $\nu = \frac{1}{m}$. For a review on the other hierarchies, see for instance[19,20,21,22].

At low energies, the effective field theory of the hierarchical FQH bulk states is described by an abelian CS gauge theory in $(1 + 2)$ dimensions. In this formulation, the fractional spin s appears as one of the parameters characterizing the topological orders in quantum Hall systems. The spin s, which arises generally from two sources, is given by the sum of two rational parameters η and θ as shown herebelow:

$$\pi s = \theta + \eta. \tag{2.1}$$

The first term, denoted by θ in eqs (2.1), comes from the Bohm-Aharonov effect due to the presence of the magnetic field B while the second parameter η is due to the Berry's phase induced by the curvature of the 2d space geometry of the system. Put differently, η is related to the curvature R of the geometry of the space; it is non zero for the two sphere S^2 whereas it vanishes on the two plane R^2 and the two Torus T^2. For more details concerning the features of the η term of eqs (2.1), see for example [11,23]. We shall focus our attention on just the first term θ by considering FQH models in $(1+2)$dimensions with a flat space geometry and study a new feature of those fractional quantum Hall excitations carrying fractional values of the spin and the charge.

A. Chern-Simons model of FQH liquids.

In the effective abelian CS gauge theory of the Haldane hierarchy of the FQH bulk states, the statistical angle θ is given by

$$\theta = \ell^T K^{-1} \ell = \sum\nolimits_i \ell^i (K^{-1})_i^j \ell_j \tag{2.2}$$

In this eq, the K_{ij} matrix has integer entries (K_{11} is odd integer while K_{ii}, $i \geq 2$, are even) and is intimately related to the filling fraction ν. The ℓ_i quantum numbers are topological charges of the elementary excitations of the FQH states. The latters are generally interpreted as quasiparticles or quasiholes, according to the sign of their electric charge $Q = -e \sum_i K_{1i} l^i$, and play an important role in the building of the Haldane hierarchy. To see the origin of the expression of the θ phase, we describe hereafter the main lines of the derivation of eq (2.1). To that purpose, consider a polarized spin layer quantum Hall system in an external electromagnetic field B of potential $A_\mu(x^0, x^1, x^2)$ with $\mu = 0; 1; 2$. The interacting

lagrangian, describing this strongly correlated electronic system in presence of B, is given by:

$$L = -eJ_\mu A^\mu \tag{2.3}$$

The effective theory of the Laughlin ground state at filling fraction $\nu = \frac{1}{m}$ is described by an abelian U(1) CS gauge theory in (1+2) dimensions of gauge field a_μ.

$$L = -[\frac{m}{2}a_\mu \partial_\nu a_\lambda + eA_\mu \partial_\nu a_\lambda]\frac{1}{2\pi}\epsilon^{\mu\nu\lambda} \tag{2.4}$$

Eq (2.4) may be obtained from eq (2.3) by using the hydrodynamic approach of the incompressible Hall liquid which takes the conserved electron current J_μ as:

$$J_\mu = \frac{1}{2\pi}\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \tag{2.5}$$

Note that the pure CS term in eq(2.4) endows each electron by m flux quanta as it may be seen from the eq of motion of the a_0 time component of the u(1) gauge field.

$$J^\mu = -\frac{e}{2\pi}\frac{1}{m}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \tag{2.6}$$

This eq describes the linear response of the ground state to the external magnetic field. Note also that up to boundary terms eq (2.4) is invariant under the U(1) gauge transformations,

$$a_\mu \to a'_\mu = a_\mu + \partial\lambda \tag{2.7}$$

A more complete effective description of the fractional quantum Hall systems is obtained by taking into account the elementary excitations effects. Introducing quasiparticle excitations of U(1) charge q in the effective theory (2.5) by inserting a source term type

$$q a_\mu J^\mu \tag{2.8}$$

and evaluating the eq of motion of the time component of the gauge field a_μ by varying eqs (2.4), one discovers the two following: (i) the electric charge Q of the excitations is fractional as shown herebelow:

$$Q = -e\frac{q}{m} \tag{2.9}$$

For $q = 1$ for instance, the fundamental quasiparticles have a fractional electric charge $-\frac{e}{m}$, carries $(\frac{1}{m})$ unit of the a_μ-flux and an induced spin $s = \frac{\theta}{\pi}$. (ii) Considering two excitations carrying q_1 and q_2 charges moving one around the other, a Bohm-Aharonov phase $\phi = 2\pi\frac{q_1 q_2}{m}$ is induced. If moreover the two excitations are identical $q_1 = q_2 = q$, one gets then a statistical angle θ given by:

$$\theta = \pi\frac{q^2}{m} \tag{2.10}$$

Note that $\frac{\theta}{\pi}$is fractional and has to be distinguished from the intrinsic spin of the electrons. It is a topological parameter induced by the presence of the B magnetic field. Note also that the fundamental quasiparticles might be viewed too simply as elementary excitations of fermions. From this view point, a bound state of m elementary excitations of spin $s = \frac{1}{m}$ and electric charge $Q = -e\frac{1}{m}$ behave exactly like a fermion of electric charge -e. The above analysis extends straightforwardly for the description of the full Haldane hierarchy based on the Laughlin ground state. At the n-th level, $n > 0$, the low energy effective theory of the Haldane hierarchy is described by an abelian $U^n(1)$ CS gauge theory in $(1+2)$dimension. The effective lagrangian of this model is given by.

$$L = -\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}[K_{ij}{a^i}_\mu \partial_\nu {a^j}_\lambda + 2eA_\mu\partial_\nu(t_i {a^i}_\lambda)] + \ell_i {a^i}_\mu J^\mu, \tag{2.11}$$

where K_{ij} are integers, K_{11} odd and K_{ii} even, characterizing the hierarchy, $t_i = \delta_{1i}$,is the so called vector charge and the ℓ_j's the number of quasiparticles of the j^{th} hierarchical level. K_{ij}, t_i and l_i are quantum numbers describing three topological orders often represented by the filling fraction ν of the quantum hall liquid, the U(1) electric charge Q and the θ statistical angle of the quasiparticles. Following the analysis of [11,20,15] by taking the matrix K as,

$$K = \begin{pmatrix} P_1 & -1 & & \\ -1 & P_2 & & \\ & & \ddots & -1 \\ & & -1 & P_n \end{pmatrix} \tag{2.12}$$

where $p_1 = m$ and $p_j = 2k_j$; j = 2,.,n, the quantities ν, Q and θ are given by:

$$\begin{aligned} \nu &= \textstyle\sum_{ij} t^i (K^{-1})_i{}^j t_j \\ Q &= -e\textstyle\sum_{ij} t^i (K^{-1})_i^j \ell_j \\ \theta &= \pi\textstyle\sum_{ij} \ell^i (K^{-1})_i^j \ell_j \end{aligned} \tag{2.13}$$

Computing the inverse of eq(2.12) and setting $t_i = \delta_{1i}$, one gets for ν and Q.

$$\begin{aligned} \nu &= K_{11}^{-1} = \frac{1}{P_1 - \frac{1}{P_2 - \frac{1}{\dots - \frac{1}{P_n}}}} \\ Q &= -e\textstyle\sum_j (K^{-1})_i^j \ell_j \end{aligned} \tag{2.14}$$

At the second level of the hierarchy one obtains, by putting n=2 in the above eqs, the following:

$$\begin{aligned} \nu &= \tfrac{P_2}{P_2 P_1 - 1} \\ Q &= -e\tfrac{P_2\ell_1 + \ell_2}{P_2 P_1 - 1} \\ \theta &= \tfrac{\pi}{P_2 P_1 - 1}[P_2 {\ell_1}^2 + \ell_1 {\ell_2}^2 + 2\ell_1\ell_2] \end{aligned} \tag{2.15}$$

In the end of this section, we would like note that in the above model, the effects of the boundary states are

ignored. A complete study of the FQH liquids should however carry these effects as it is also required by experiments which reveal the existence of edge excitations with finite velocities. In the next section, we show how these effects may be incorporated. Later on we also examine their couplings.

B. Edge excitations

The edge excitations of the FQH liquids are conveniently described by a 2d boundary conformal field theory whose action may be obtained by evaluating eq(2.4) on the boundary of the system. The result one gets is:

$$S_{edge} = \frac{1}{4\pi}\int_{\partial M} dtdx[K_{ij}\partial_0\phi^i\partial_x\phi^j - V_{ij}\partial_x\phi^i\partial_x\phi^j]. \tag{2.16}$$

To establish this relation, let us introduce some convention notations. (a) We denote by M the (1 + 2) dimensional space-time, parameterized by the local coordinates (x^0, x^1, x^2) in which the Chern-Simons gauge fields a_μ live and by ∂M its boundary. (b) We write M as $M = R \times \Sigma$ where Σ is the subset parameterized by the space variable x^1 and x^2, representing the surface where evolve the electrons of the FQH liquid. In this case, ∂M is given by

$$\partial M = R \times \partial\Sigma \tag{2.17}$$

where $\partial\Sigma$ stands for the one dimensional border of Σ. To get the boundary effective theory of the FQH states with finite velocities, one has to work a little bit hard. First one should note that the pure Chern-Simons gauge action,

$$S = -\frac{1}{4\pi}\int_M {a_\mu}^i[K_{ij}\partial_\nu {a_\rho}^j\epsilon^{\mu\nu\rho}] \tag{2.18}$$

is, in general, not invariant under the gauge transformation of the CS gauge fields ${a_\mu}^i$:

$${a'_\mu}^i = {a_\mu}^i + \partial_\mu\lambda^i; i = 1, ..., n, \tag{2.19}$$

where $\lambda^i = \lambda^i(x^0, x^1, x^2)$ are gauge parameters. Putting the change (2.19) into eq (2.16), one obtains the following variation of the gauge action

$$\Delta S = -\frac{1}{2\pi}\int_M \partial_\mu[K_{ij}\lambda^i\partial_\nu {a_\rho}^j\epsilon^{\mu\nu\rho}] \tag{2.20}$$

Integrating eq (2.20) by using Stokes theorem, we get by using eq (2.17)

$$\Delta S = \frac{1}{2\pi}\int_{\partial M} K_{ij}\lambda^i[\partial_0 {a_t}^j - \partial_t {a_0}^j] \tag{2.21}$$

where the index t refers to the tangent direction of $\partial\Sigma$. In general eq (2.21) is non zero. To restore the gauge invariance of the action S eq(2.16), we require that gauge parameters $\lambda^i(x^0, x^1, x^2)$ vanish on $\partial M = R \times \partial\Sigma$. In other words we demand the following:

$$\lambda(x^0, \{x^1, x^2\} \in \partial\Sigma) = 0 \tag{2.22}$$

However there is a price one should pay for this choice since due to the restriction (2.22), some degrees of freedom of the ${a_\mu}^i$ gauge fields on the boundary become dynamical. These degrees of freedom give us the boundary effects we are looking for. To describe the dynamics of the boundary gauge degrees of freedom, we need to specify a gauge fixing condition. One way to do it is to take ${a_0}^i = 0$ and interpret the bulk equations of the motion for ${a_0}^i$ as constraint eqs. Thus one remains with the eqs of motion of ${a_j}^i$ which read as:

$$\epsilon^{0\alpha\beta}[K_{ij}\partial_\alpha {a_\beta}^j] = 0 \tag{2.23}$$

Eq (2.23) is solved by introducing a set of two dimensional $\{\phi^i\}$ and taking the space components of the gauge fields${a_\mu}^i$ as:

$${a_\mathbf{j}}^i = \partial_\mathbf{j}\phi^i \tag{2.24}$$

Putting back this solution into eq (2.16) and again integrating by parts, one finds after using the choice ${a_0}^i = 0$, the following:

$$S_{edge} = \frac{1}{4\pi}\int_{\partial M} K_{ij}\partial_0\phi^i\partial_t\phi^j. \tag{2.25}$$

This is not the full story since the action we have obtained has zero energy and so does not describe the boundary effects. In fact, eq (2.25) recovers just a part of the result as it may be seen on eq(2.16). This ambiguity was expected because we knew already that edge excitations have finite velocities, a feature which has been ignored in the CS formulation of the FQH effective theory. To solve this problem, we have to find a way to insert the edge excitations velocities in the effective theory. A tricky way to do it is to put the velocities through the gauge fixing condition. For the simplicity of the presentation, we prefer to give hereafter a sketch of the method for the level one of the hierarchy where the $\mathbf{n} \times \mathbf{n}$ matrix K_{ij} reduces to the integer P_1 which we set m. This method is achieved in two steps: First make a special SO(1.2) Lorentz transformation on the $\{x^0, x^1\}$ plane coordinates of the system $\{x^\mu\}$

$$\begin{array}{ll} y^0 = x^0 & ; \partial/\partial y^0 = \partial/\partial x^0 + v\partial/\partial x^1 \\ y^1 = x^1 - vx^0 & ; \partial/\partial y^1 = \partial/\partial x^1 \\ y^2 = x^2 & ; \partial/\partial y^2 = \partial/\partial x^2 \end{array} \tag{2.26}$$

where v is the velocity of the particle. Under this change, the gauge field a_μ also transforms as

$$\begin{array}{l} b^0 = a^0 + va^1 \\ b^1 = a^1 \\ b^2 = a^2 \end{array} \tag{2.27}$$

while the pure U(1) CS action remains invariant as shown herebelow.

$$\begin{aligned} S &= \tfrac{m}{4\pi}\int_M d^3x a_\mu \partial_\nu a_\rho \epsilon^{\mu\nu\rho} \\ &= \tfrac{m}{4\pi}\int_M d^3y b_\mu \partial_\nu b_\rho \epsilon^{\mu\nu\rho} \end{aligned} \tag{2.28}$$

The second step is to take $b_0 = 0$; that is $a_0 = -va_1$ instead of a_0=0 as done for deriving eq(2.26). Then integrating by part one finds, after renaming the tangent coordinate of $\partial\Sigma$ as x and its normal as y and using eqs(2.26), the following action

$$S = \frac{1}{4\pi}\int_{\partial M} dtdx[m\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} + mv(\frac{\partial\phi}{\partial x})^2] \tag{2.29}$$

Eq (2.29) shows that the second term is just minus the hamiltonian of the edge excitation states and so the product mv should be negative definite. In other words consistency of the boundary effective theory requires v and m to have opposite signs. For a level $\mathbf{n}$ of the Haldane hierarchy, the effective edge theory is just an extension of the models (2.28) given by the action (2.16) where K_{ij} is a $\mathbf{n}\times\mathbf{n}$ integer matrix defining the hierarchy and V_{ij} is a positive definite matrix carrying the boundary effect of the FQH liquids. For a more rigorous derivation of the boundary effective action (2.16), see for instance [11,24].

III. FRACTIONAL QUANTUM HALL DROPLET APPROACH

So far we have learned that an abelian FQH system can be described by the CS gauge theory eq(2.16) where the symmetric matrix K defines topological order parameters characterizing the internal structure of the system [25,26]. In the Haldane construction on the Laughlin ground state, with filling fraction $\frac{1}{m}$, the electric charge q_l of quasiparticles, the statistical angle θ and the filling fraction of the Hierarchical system are all of them expressed in terms of K, the charge vector t and the l_i's quantum numbers. Up on making an assumption on the geometry of the FQH system approximating the liquid to droplets, one can make several theoretical predictions by using the power of the methods of CFT.

Here we want to use the drolplet approach [27] of the FQHE in order to: (i) develop an explicit derivation of a receipt giving a realisation of hierarchical states at second level in terms of composites of two Laughlin states. To our knowledge this special representation which is often used in FQHE literature has never been prooved rigorously. Here we not only answer this question but we give also its extension to any level k of the Haldane hierarchy. More precisely we show, by using an appropriate $GL(k,Z)$ change of CS gauge variables, going beyond the $SL(k,Z)$ symmety preserving the global property of the hypercubic Z^k charge lattice, that we can usually interpret a generic level k Haldane state of filling fraction ν_H as a composite of k Laughlin states of filling fractions ν_I; $I = 1,\ldots,k$. (ii) analyse the interactions of the various droplet branch edge excitations by using the special representation we refer to in the first point(i); see also eq(5.6) and the subsequent one. these couplings, which are carried by a symmetric and positive matrix V_{ij}, may be integrated out by making an appropriate choice of the CS gauge field basis diagonalising the matrix V_{ij}. Like for known results based on the representation(3.13), here also we show that the full hamiltonian is diagonalised by introducing effective velocities. (iii) build the wave functions of quasiparticles and explore further the property of fractionality of their spins and electric charges. These wave functions will be interpreted in section 7 as HW states of RdTS representations.

A. Haldane state as composites of Laughlin states

To start, consider the effective bulk theory at a level k of Haldane hierarchy described by CS gauge theory(2.11). Under a specific linear combination of the CS gauge fields a_μ i.e,

$$\tilde{a}_{i\mu} = U_{ij}a_{j\mu}, \tag{3.1}$$

where U_{ij} is an invertible $k\times k$ matrix preserving the charge quantization. U_{ij} will be specified explicitly on the levels $k=2$ and $k=3$ examples. Note that the U_{ij} matrices we will be considering here belong to $Gl(k,Z)$. They are not symmetries of the CS gauge action (2.11) nor even of the Z^k charge lattice in which the quantum topological vector t takes its values. They are rather symmetries of the filling fraction ν_H eqs(2.13). Using the above transformation, one can diagonalise the K matrix so that the effective lagrangian(2.11) is now replaced by the new following one:

$$L = -\frac{1}{4\pi}\sum_i(\tilde{a}_{i\mu}D_{ii}\partial_\nu\tilde{a}_{i\lambda}\epsilon^{\mu\nu\lambda} - \frac{e}{2\pi}T_iA_\mu\partial_\nu\tilde{a}_{i\lambda}\epsilon^{\mu\nu\lambda}) \tag{3.2}$$

where $D = U^tKU$. Eq(3.2) defines a system of k free abelian CS gauge theories, where each one describes an effective model of a Laughlin state of filling fraction $\nu_i = {T_i}^2(D^{-1})_{ii}$. Integrating out the time component of the gauge fileds $\tilde{a}_{i0}$ by using equation of motion $\frac{\delta L}{\tilde{a}_{i0}} = 0$, we find the following filling fraction ν_H of the new system:

$$\nu_H = \sum_i {T_i}^2(D^{-1})_{ii} \tag{3.3}$$

where the index H refers to Haldane. Now imposing invariance of ν_H under the $Gl(k,Z)$ transformation,i.e;

$$\nu_H = \sum_i {T_i}^2(D^{-1})_{ii} = \sum_{ij} t^i{(K^{-1})_i}^j t_j \tag{3.4}$$

and choosing the T vector charge as $T = (1,\ldots,1)$, one can determine explicitly the form of the U matrix of

eq(3.1). Note in passing that the above eq may be interpreted as describing the ν_H state containing k components of incompressible fluids. Each component ν_i, $i = 1, 2, \ldots, k$, corresponds to a Laughlin state with filling fraction $\nu_j = \frac{1}{2m_j+1}$, m_j integer, which in the hydrodynamical approach is associated to a branch described by a c=1 free boundary CFT [11]. This interpretation is renforced by computing the total electric charge and the spin of the full state. Mimicking the analysis of section 2, we find after a straightforward calculations the two following:

$$q_\ell = -eTD^{-1}\ell \tag{3.5}$$

where $\ell = (\ell_1 ... \ell_k)$ is the toplogical charge of the elementary excitations at a X_0 position. Similarly we find for the spin:

$$\frac{\theta}{\pi} = \ell D^{-1}\ell \tag{3.6}$$

To illustrate the idea let us give two examples; the first one concerns a state at the second level of the Haldane hierarchy (k=2) and the other one concerns a state at the third level of the hierarchy(k=3). After these examples, we also give the general result we have obtained.

(a)$\nu_H = \frac{2}{5}$

Here we consider the FQH state with filling fraction $\nu_H = \frac{2}{5}$; it has been studied from different views using too particularly Haldane, Halperin and Jain series [13,20]. In Haldane hierarchy, the $K_{\frac{2}{5}}$ matrix is:

$$K_{\frac{2}{5}} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Next, choosing the U transformation matrix eq(5.1) as:

$$U_{\frac{2}{5}} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

we get the following diagonal matrix:

$$D_{\frac{2}{5}} = \begin{pmatrix} 3 & 0 \\ 0 & 15 \end{pmatrix}$$

So ν_H takes the remarkable decomposition, $\nu_H = \frac{1}{3} + \frac{1}{15}$, which may viewed as a composite configuration of two Laughlin fundamental states. This special feature is renforced by the computation of the total electric charge $q_{(\ell_1,\ell_2)}$ and the total statistics angle $s_{(\ell_1,\ell_2)}$. For the electric charge we have the following:

$$q_{(\ell_1,\ell_2)} = -e(\frac{\ell_1}{3} + \frac{\ell_2}{15}).$$

Similarly we have for the spin $s_{(\ell_1,\ell_2)} = \frac{\theta}{\pi}$,

$$s_{(\ell_1,\ell_2)} = (\frac{{\ell_1}^2}{3} + \frac{{\ell_2}^2}{15}).$$

Therefore the 2/5 FQH state may be interpreted as consisting of two free components of the incompressible fluid. The 2/5 FQH excitations consist then of two kinds of quasiparticles respectively given by the excitations of 1/3 and 1/15 Laughlin states. This analysis may be easily extended to the case of higher levels of the hierarchy as shown on the following example.

(b)$\nu_H = \frac{3}{7}$

For the case of the third hierarchical level at filling fraction $\nu_H = \frac{3}{7}$, where the corresponding $K_{\frac{3}{7}}$ matrix is:

$$K_{\frac{3}{7}} = \begin{pmatrix} 3 & -1 & o \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Moreover using the $U_{\frac{3}{7}}$ matrix given by:

$$U_{\frac{3}{7}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ 1 & -2 & -2 \end{pmatrix}$$

we get the following:

$$D_{\frac{3}{7}} = \begin{pmatrix} 3 & & \\ & 15 & \\ & & 35 \end{pmatrix}$$

As discussed above for the level k=2 case, we have here also for k=3;

$$\nu_H = \frac{1}{3} + \frac{1}{15} + \frac{1}{35},$$

$$q_{(\ell_1,\ell_2,\ell_3)} = -e(\frac{\ell_1}{3} + \frac{\ell_2}{15} + \frac{\ell_3}{35}),$$

$$s_{(\ell_1,\ell_2,\ell_3)} = (\frac{{\ell_1}^2}{3} + \frac{{\ell_2}^2}{15} + \frac{{\ell_3}^2}{35}).$$

In summary the $\frac{3}{7}$ FQH state may be interpreted as a composite state consisting of three branches with same propagating modes sign. We shall return to study aspects of these propagating modes in a moment; for the time being let us make two remarks. First note that the $\frac{3}{7}$ FQH state can be also viewed as a composite of the $\frac{2}{5}$ and the $\frac{1}{35}$ FQH states in agreement with associativity property of the tensor product of Hilbert space wave functions. Second such analysis is a priori extendable to higher orders of hierarchy. We have checked explicitly this feature for the leading terms of the Haldane series and extended it to any generic level k of the hierarchy using a special property of the continuous fraction eq(2.14). Indeed we have shown that (2.14) can be usually put in the following remarkable form:

$$\nu_H = \sum_{j=0}^{k-1} \frac{1}{m_j m_{j+1}} \tag{3.7}$$

where $m_j = p_j m_{j-1} - m_{j-2}$; with $m_0 = 1$ and $m_1 = p_1$ and where the p_j's are as in eq (2.14). Note that the $m_j m_{j+1}$ product is usually a odd integer. Since m_1 is odd it follows then all the m_j's are odd and so the generic terms $\nu_j = \frac{1}{m_j m_{j+1}}$ may be interpreted as filling fractions of laughlin states. From this eq, it is not difficult to derive the $D^k_{\nu_H}$ which reads as:

$$D^{(k)}_{\nu_H} = \begin{pmatrix} m_1 & & & \\ & m_1 m_2 & & \\ & & \ddots & \\ & & & m_{k-1} m_k \end{pmatrix} \tag{3.8}$$

This eq establishes the extention to any value of k, the result we have illustrated on the $\nu_H = \frac{2}{5}$ and $\nu_H = \frac{3}{7}$ examples. It shows clearly that a generic state at level k of the Haldane hierachy may usually thought of as a composite of k Laughlin states with different positive modes of branches. Note in passing that our analysis is general as it also valid for branches with opposite modes such as in the example $\nu_H = \frac{2}{3}$ state belonging to the Haldane second level series by taking p_2 an even negative integer. Knowing D_{ν_H}, one may also derive the $GL(k, Z)$ U-transformation by help of eq(2.12). Indeed starting from the constraint eqs,

$$(D_{\nu_H})_{ij} = U_{ni} (K_{\nu_H})_{nm} U_{mj} \tag{3.9}$$

where D_{ij} and K_{nm} are respectively given by eqs (3.8) and (2.12) and where U_{mj} are choosen as follows:

$$U_{\nu_H} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kk} \end{pmatrix} \tag{3.10}$$

where b_{ij} are integers to be determined. These eqs are non linear numerical ones not easy to solve in general. In case of Haldane hierarchy at the second level, eqs (3.9) has three solutions given by:

$$\begin{aligned} U_{(I)} &= \begin{pmatrix} 1 & 1 \\ 0 & m_1 \end{pmatrix}; \\ U_{(II)} &= \begin{pmatrix} 1 & 1 \\ \frac{2m_1}{m_2+1} & m_1 (\frac{1-m_2}{1+m_2}) \end{pmatrix}; \\ U_{(III)} &= \begin{pmatrix} 1 & 1 \\ \frac{2m_1}{m_2+1} & m_1 \end{pmatrix} \end{aligned} \tag{3.11}$$

These solutions lead to the same $D^{(2)}_{\nu_H}$ matrix

$$D^{(2)}_{\nu_H} = \begin{pmatrix} m_1 & 0 \\ 0 & m_1 m_2 \end{pmatrix} \tag{3.12}$$

but not all of them belong to $GL(2, Z)$. If, in addition to U_I, we require that the two other transformations also belong to $Gl(2, Z)$; that is $l(m_2 + 1) = m_1$ for integer l's, we get constraints on the FQH filling fractions. In this case the three $U_I, U_{II} and U_{III}$ transformations are all of them in $Gl(2, Z)$ and have a remarkable interpretations. They lead to stable states associated with the observable filling fractions as $v = 2/3, 2/5, 2/9, ...$ For $l(m_2 + 1) \neq m_1$, $U_{II} and U_{III}$ are no loger acceptable since only U_I is in $Gl(2, Z)$. At first sight, the ν_H solutions allowed by $Gl(2, Z)$ turn out to be associated with unstable states. We still do not understand this property.
For level three of the hierarchy, eqs (3.9) read as:

$$\begin{aligned} m_1 - 2a + a^2 \tfrac{m_2+1}{m_1} - 2ad + d^2 \tfrac{m_3+m_1}{m_2} &= m_1 \\ m_1 - 2b + b^2 \tfrac{m_2+1}{m_1} - 2be + e^2 \tfrac{m_3+m_1}{m_2} &= m_1 m_2 \\ m_1 - 2c + c^2 \tfrac{m_2+1}{m_1} - 2cf + f^2 \tfrac{m_3+m_1}{m_2} &= m_2 m_3 \\ m_1 - (a + b) + ab \tfrac{m_2+1}{m_1} - ea - bd + ed \tfrac{m_3+m_1}{m_2} &= 0 \\ m_1 - (a + c) + ac \tfrac{m_2+1}{m_1} - fa - cd + fd \tfrac{m_3+m_1}{m_2} &= 0 \\ m_1 - (b + c) + bc \tfrac{m_2+1}{m_1} - ce - fb + ef \tfrac{m_3+m_1}{m_2} &= 0 \end{aligned} \tag{3.13}$$

where we have taken U_{ij} as,

$$U_{\nu_H} = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{pmatrix} \tag{3.14}$$

Here also we have several solutions, an acceptable one of them was already encountered previously eq (3.13); an other one is given by the following triangular matrix.

$$U_{\nu_H} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & m_1 & m_1 \\ 0 & 0 & m_2 \end{pmatrix} \tag{3.15}$$

The above calculations may be extended for higher hierarchical levels k ; the number of solutions rise assymptoticaly with the level value of k. A particular solution for the generic case was already identified, but still needs more analysis. Details on this issue as well as branches with opposite modes and more informations on the $Gl(k, Z)$ transformations will be reported elsewhere [27].

B. Dynamics of droplets edge excitations

To start recall that in the droplet approximation, droplet waves are identified as edge excitations of FQH fluid. These waves move along the edge of the sample, considered here as a finite FQH liquid with filling fraction $\nu = \frac{1}{m}$ confined by a smooth potential well, in which the electrons propagate with the velocity $v = \frac{E}{B} c$ [28].
In the language of 2d CFT, edge excitations of FQH liquid with k branches are described a c=k CFT theory with a $U(1)^k$ Kac-Moody symmetry. For the leading case, the corresponding c=1 CFT describes a Laughlin state having one branch edge excitations and a disk like geometry. The particle density ρ is related to the bosonic field ϕ as $\rho(x) = \frac{1}{2\pi} \partial_x \phi$; this is just the electron density operator on the edge which may be thought of as the U(1)

Kac-Moody current. Expanding this current in Laurent series in terms of the Laurent modes ρ_k and using the U(1) Kac-Moody algebra,

$$[\rho_k, \rho_{k\prime}] = \frac{\nu}{2\pi} k\delta_{k+k\prime} \tag{3.16}$$

one can show that the hamiltonian H giving the quantum energy of the edge excitations is proportional to the velocity v and reads in terms of the ρ_k modes as:

$$\begin{array}{rcl} H &=& -2\pi\sum_{k>0} v\rho_k\rho_{-k}, \\ [H, \rho_k] &=& -vk\rho_k \end{array} \tag{3.17}$$

Charged excitations are created by the following vertex field operator ψ;

$$\psi \propto e^{-\frac{1}{\sqrt{\nu}}\phi} \tag{3.18}$$

Moreover using the fact that in this CFT, ϕ is just a free phonon field having a two point function $< \phi(z_1)\phi(z_2) >\sim lnz_{12}$, one can easily establish that the ψ field operator carries a unit electric charge and a conformal spin $s = \frac{1}{2\nu}$ as shown on the following commutation relations.

$$\begin{array}{rcl} [\rho(x), \psi(x\prime)] &=& \delta(x - x\prime)\psi(x\prime) \\ [L_0, \psi(x)] &=& \frac{1}{2\nu}\psi(x). \end{array} \tag{3.19}$$

Similar calculations shows moreover that the propagator of the ψ field operator is that of a chiral Luttinger liquid [29].

$$G(x,t) \propto \frac{1}{(x - vt)^{\frac{1}{\nu}}}.$$

Hierarchical states of level k contain several component of the incompressible fluid, each component give rise to a branch of the edge excitations and so are described by a c=k CFT with $U^k(1)$ Kac Moody symmetry.

Extending the previous analysis to generic k values, we get the following relations regarding the $\rho_{i,k}$ field operators.

$$\begin{array}{rcl} [\rho_{i,k}, \rho_{i,k\prime}] &=& T_i(D^{-1})_{i,j}\frac{1}{2\pi}k\delta_{k+k\prime}, \\ \rho_e &=& -e\sum_i \rho_i, \\ [\rho(x), \psi(x\prime)] &=& \ell_i(D^{-1})_{ij}\delta(x - x\prime)\psi_\ell(x\prime), \\ \psi_\ell &\propto& e^{i\frac{\ell_i}{T_i}\phi_i} \end{array} \tag{3.20}$$

In these eqs, the ϕ_i's are the bosonic scalars associated with each branch of the fluid while ρ_e is the total electron density. An electron excitation appears whenever $\sum_i (D^{-1})_{ii}\ell_i = 1$ is fulfilled; this implies in turn that $\ell_i = \sum_i D_{ij}L_j$ with $\sum_i L_i = 1$. The electron operator ψ_{eL} takes then the form:

$$\psi_{e,L} \propto e^{i\sum_i \frac{\ell_i}{T_i}\phi_i} \tag{3.21}$$

and obeys the following anticommutation relation:

$$\psi_{eL}(x)\psi_{eL}(x\prime) = (-1)^\lambda\psi_{eL}(x\prime)\psi_{eL}(x) \tag{3.22}$$

where $\lambda = \sum_i L_i\nu_i^{-1}L_i$; i.e an odd integer. Before going ahead note that each factor $e^{i\frac{\ell_i}{T_i}\phi_i}$ in eq(3.20) defines the wave function of a quasiparticle of fractional spin. If one forgets about FQHE for a wile and just retains that on $\partial(M)$ lives a conformal structure, one may consider its highest weight representations which read in general as:

$$\begin{array}{l} L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle, \\ L_n|h, \bar{h}\rangle = 0; \quad n \geq 1 \\ \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle, \\ \bar{L}_n|h, \bar{h}\rangle = 0; \quad n\geq 1, \\ cI|h, \bar{h}\rangle = c|h, \bar{h}\rangle \end{array} \tag{3.23}$$

where $|h, \bar{h}\rangle$ are Virasoro primary states. To make contact with the FQH quantities we have been considering, we give the following correspondence. (i) the vertex operator $e^{i\frac{\ell_i}{T_i}\phi_i}$ may be represented generally by a 2d field operator $\phi_{h,\bar{h}}(z, \bar{z})$. This is a primary conformal field representation of conformal scale δ= h+ $\bar{h}$ and conformal spin s= h- $\bar{h}$. In the present case the primary fields are chiral and so $\bar{h} = 0$. (ii) The highest weight states $|s\rangle$ associated with the chiral fields $\phi_s(z)$ are related as $\Phi_s(0)|0\rangle = |s\rangle$ where now the spin s is given by eqs(3.9) and (3.12).

C. Branch Interactions

Recall that in absence of interactions and ignoring the edge excitations velocities v_I, the action of the system reads as,

$$S = -\frac{1}{4\pi}\int d^3x D_{ii}\tilde{a}_{i\mu}\partial_\nu\tilde{a}_{i\lambda}\epsilon^{\mu\nu\lambda} \tag{3.24}$$

On the n branches of the droplet, the dynamics of the excitations are conveniently described by a boundary c=n CFT. In what follows we use the representation (3.19) in order to study the branch interactions in presence of velocities v_i. We will show that, up on redefining the Kac-Moody currents ρ_i, the FQH droplet is still described by a c=n CFT provided replacing the v_i's by new velocities $\tilde{v_i}$. To that purpose we will start first by introducing the v_i velocities in the system. Using the change(3.1-9) and following the same calculus we have developed in section 2 by taking $\tilde{a}_i^0 = -v_i\tilde{a}_i^1$, we find, after performing an integration by part, the following edge action:

$$S_{edge} = \frac{1}{4\pi}\int dxdt[D_{ii}\partial_t\phi^i\partial_x\phi^i + v_iD_{ii}\partial_x\phi^i\partial_x\phi^i] \tag{3.25}$$

The edge dynamics is now described by the following hamiltonian generalising eq(3.17);

$$H = -\frac{1}{2\pi}\sum_i v_iD_{ii}\partial_x\phi^i\partial_x\phi^i \tag{3.26}$$

In this formalism we have here also assumed a large gap between different edges; i.e the radii r_i and r_j of two droplets are such that: $r_i - r_j \gg \ell_o^2$ where ℓ_o is the magnetic lenght. If now we take edges interactions into account, the above Hamiltonian becomes:

$$H = \frac{-1}{2\pi}\sum_i V_{ij}\partial_x\phi^i\partial_x\phi^j \tag{3.27}$$

where the symmetric $k \times k$ matrix V_{ij} contains non diagonal terms in addition to the diagonal ones eq(3.26). A way to deal with this hamiltonian is to diagonalise V_{ij} by performing a transformation on the Kac-Moody currents $\partial_x\phi^i$ and keeping D_ν diagonal. This gives the new velocities $\tilde{v}_i$. Thus, under a linear trasformation W, which act on ρ_i as,

$$\rho_i = W_{ij}\tilde{\rho}_j \tag{3.28}$$

we should have:

$$W^t D W = D'$$

$$W^t V W = \lambda_i \delta_{ij}$$

where λ_i are real parameters defining the new velocities $\tilde{v}_i$. To illustrate the method, let us consider the case of FQH system with two branches and an interacting matrix.

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}.$$

Under the particular transformation W,

$$\begin{aligned} \rho_1 &= \tfrac{1}{\sqrt{B_1}} cos\alpha\tilde{\rho}_1 + \tfrac{1}{\sqrt{B_1}} sin\alpha\tilde{\rho}_2 \\ \rho_2 &= -\tfrac{1}{\sqrt{B_2}} sin\alpha\tilde{\rho}_1 + \tfrac{1}{\sqrt{B_2}} cos\alpha\tilde{\rho}_2 \end{aligned} \tag{3.29}$$

where $B_I = T_I \frac{v_I}{\nu_I}$, the V potential matrix is diagonalized for α angles constrained as:

$$tg2\alpha = -2\frac{V_{12}\sqrt{B_1 B_2}}{B_2 V_{11} - B_1 V_{22}} \tag{3.30}$$

The new velocities are then,

$$\begin{aligned} \tilde{v}_1 &= \tfrac{V_{11}}{B_1}\tfrac{cos^4\alpha}{cos2\alpha} - \tfrac{V_{22}}{B_2}\tfrac{sin^4\alpha}{cos2\alpha} \\ \tilde{v}_2 &= -\tfrac{V_{11}}{B_1}\tfrac{sin^4\alpha}{cos2\alpha} + \tfrac{V_{22}}{B_2}\tfrac{cos^4\alpha}{cos2\alpha} \end{aligned} \tag{3.31}$$

Finally, the diagonalised hamiltonian in terms of the new variables as,

$$H = -\frac{1}{4\pi}\sum_i \tilde{v}_i\partial_x\tilde{\phi}^i\partial_x\tilde{\phi}^i \tag{3.32}$$

and describes indeed a c=n boundary CFT. Note in the end of this paragraph that W taken here is a $O(k,R)$ transformation and so affects the level of the $U(1)^k$ Kac Moody algebra eq(3.19). Under the linear transformation W choosen as in eqs(3.28-29), the new current operators $\tilde{\rho}_i$ generate a $U(1)^k$ Kac Moody algebra of level one rather than a level ν as in eq(3.19). Setting $B_1 = 1$ and $B_2 = 1$, the transfomation W becomes orthogonal and so ρ_i and $\tilde{\rho}_i$ obey the same algebra.

IV. RDTS SUPERSYMMETRY

RdTS supersymmetry is a special generalisation of fractional supersymmetry in two dimensions which was considered in many occasions in the past in connection with integrable deformations of conformal invariance and representations of the universal envelopping $U_q sl(2)$ quantum ordinary and affine symmetries[30,3,4]. Like for fsusy, highest weight representations of RdTS algebra carry fractional values of the spin and obey more a less quite similar eqs. Both fsusy and RdTS invariances describe residual symmetries which are left after integrable deformations of infinite dimensional invariances by relevant operators. We propose to describe hereafter the main lines of RdTS invariance by considering a (1+2) dimensional space time M with a boundary ∂M.

A. Extension of the $\mathbf{P_{(1,2)}}$ Poincaré algebra

To start consider the Poincaré symmetry of the $R^{1,2}$ space generated by the space time translations P_μ and the Lorentz rotations J_α satisfying altogether the following closed commutation relations:

$$\begin{aligned} &[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta}\eta^{\gamma\delta}J_\delta \\ &[J_\alpha, P_\beta] = i\epsilon_{\alpha\beta\gamma}\eta^{\gamma\delta}P_\delta \\ &[P_\mu, P_\nu] = 0 \end{aligned} \tag{4.1}$$

In these eqs, $\eta_{\alpha\beta} = diag(1,-1,-1)$ is the $R^{1,2}$ Minkowski metric and $\epsilon_{\alpha\beta\gamma}$ is the completly antisymmetric Levi-Civita tensor such that $\epsilon_{012} = 1$. A convenient way to handle eqs(4.1) is to work with an equivalent formulation using the following Cartan basis of generators $P_\mp = P_1 \pm iP_2$ and $J_\mp = J_1 \pm iJ_2$. In this basis eqs(4.1) read as:

$$\begin{aligned} [J_+, J_-] &= -2J_0 \\ [J_0, J_\pm] &= \pm J_\pm \\ [J_\pm, P_\mp] &= \pm P_0 \\ [J_+, P_+] &= [J_-, P_-] = 0 \\ [J_0, P_0] &= [P_\pm, P_\mp] = 0 \end{aligned} \tag{4.2}$$

The algebra (4.1-2) has two Casimir operators $P^2 = {P_0}^2 - \frac{1}{2}(P_+P_- + P_-P_+)$ and $P.J = P_0J_0 - \frac{1}{2}(P_+J_- + P_-J_+)$. When acting on highest weight states of mass m and spin s, the eigenvalues of these operators are m^2 and ms respectively. For a given s, one distinguishes two classes of irreducible representations: massive and massless representations. To build the $so(1,2)$ massive representations, it is convenient to go to the rest frame where the momentum vector P_μ is $(m,0,0)$ and the $SO(1,2)$ group reduces to its abelian $SO(2)$ little subgroup generated by J_0; $(J_\pm = 0)$. In this case, massive irreducible representations are one dimensional and are parametrized by a real parameter. For the full $SO(1,2)$

group however, the representations are either finite dimensional for $|s| \in \mathbf{Z}^{+}/2$ or infinite dimensional for the remaning values of s.
Given a primary state $|s\rangle$ of spin s, and using the abovementioned $SO(1,2)$ group theoretical properties, one may construct in general two representations HWR(I) and HWR(II) out of this state $|s\rangle$. The first representation HWR(I) is given by:

$$\begin{aligned}
J^0|s\rangle &= s|s\rangle \\
J_-|s\rangle &= 0 \\
|s,n\rangle &= \sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(J_+)^n|s\rangle, n\geq 1 \qquad (4.3) \\
J_0|s,n\rangle &= (s+n)|s,n\rangle \\
J_+|s,n\rangle &= \sqrt{(2s+n)(n+1)}|s,n+1\rangle \\
J_-|s,n\rangle &= \sqrt{(2s+n-1)n}|s,n-1\rangle
\end{aligned}$$

The second representation HWR(II) is defined as:

$$\begin{aligned}
\bar{J}_0|\bar{s}\rangle &= -s|\bar{s}\rangle \\
\bar{J}_+|\bar{s}\rangle &= 0 \\
|\bar{s},n\rangle &= (-)^n\sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(\bar{J}_-)^n|\bar{s}\rangle \qquad (4.4) \\
\bar{J}_0|\bar{s},n\rangle &= -(s+n)|\bar{s},n\rangle \\
\bar{J}_+|\bar{s},n\rangle &= -\sqrt{(2s+n-1)(n)}|\bar{s},n+1\rangle
\end{aligned}$$

Note in passing that in the second module we have supplemented the generators and the representations states with a bar index. Note moreover that both HWR(I) and HWR(II) representations have the same $so(1,2)$ Casimir $C_s = s(s-1), s<0$. For $s \in \mathbf{Z}^-/\mathbf{2}$, these representations are finite dimensional and their dimension is $(2|s|+1)$. For generic real values of s, the dimension of the representations is however infinite. If one chooses a fractional value of s say $s=-\frac{1}{k}$; each of the two representations (4.3-4) splits a priori into two isomorphic representations respectively denoted as $D^+_{\pm 1/k} and D^-_{\pm 1/k}$. This degeneracy is due to the redundancy in choosing the spin structure of $\sqrt{-2/k}$ which can be taken either as $+i\sqrt{2/k}$ or $-i\sqrt{2/k}$. These representations are not independent since they are related by conjugations; this why we shall use hereafter the choice of [1] by considering only $D^+_{-1/k}$ and $D^-_{-1/k}$. In this case the two representation generators $J_{0,\pm}$ and $\bar{J}_{0,\pm}$ are related as:

$$\bar{J}_{0,\mp} = (J_{0,\pm})^* \qquad (4.5)$$

Furthermore taking the tensor product of the primary states $|s\rangle$ and $|\bar{s}\rangle$ of the two $so(1,2)$ modules HWR(I) and HWR(II) and using eqs(4.3-4), it is straightforward to check that it behaves like a scalar under the full charge operator $J_0 + \bar{J}_0$:

$$(J_0 + \bar{J}_0)|s\rangle \otimes |\bar{s}\rangle = 0 \qquad (4.6)$$

Eq(4.6) is a familiar relation in the study of primary states of Virasoro algebra. This equation together with the mode operators J^n_- and $\bar{J}^m_+$ which act on $|s\rangle \otimes |\bar{s}\rangle$ as:

$$\begin{aligned}
(J_-)^n|s\rangle \otimes |\bar{s}\rangle &= 0, \quad n\geq 1 \\
(\bar{J}_+)^m|s\rangle \otimes |\bar{s}\rangle &= 0, \quad m\geq 1 \qquad (4.7)
\end{aligned}$$

define a highest weight state which looks like a Virasoro primary state of spin 2s and scale dimension $\delta = 0$. In [8], we have shown that eqs(4.6-7) are indeed related to HWR of conformal invariance which is written as:

$$\begin{aligned}
(L_0 - \bar{L}_0)\Phi_{h,\bar{h}}(0,0)|0\rangle &= (h-\bar{h})\Phi_{h,\bar{h}}(0,0)|0\rangle \\
(L_0 + \bar{L}_0)\Phi_{h,\bar{h}}(0,0)|0\rangle &= (h+\bar{h})\Phi_{h,\bar{h}}(0,0)|0\rangle \qquad (4.8) \\
L_n\Phi_{h,\bar{h}}(0,0)|0\rangle &= 0 \\
\bar{L}_m\Phi_{h,\bar{h}}(0,0)|0\rangle &= 0
\end{aligned}$$

where L_n and $\bar{L}_m$ are respectively the usual left and right Virasoro modes and $\Phi_{h,\bar{h}}(z,\bar{z})$ is a primary conformal field representation of conformal scale $h+\bar{h}$ and conformal spin $h-\bar{h}$. The primary $so(1,2)$ highest weight states $|s\rangle$ and $|\bar{s}\rangle$ eqs(4.3-4) are respectively in one to one correspondance with the left Virasoro primary state $\Phi_h(0)|0\rangle = |h\rangle$ and the right Virasoro primary one $\Phi_{\bar{h}}(0)|0\rangle = |\bar{h}\rangle$. On the other hand, if we respectively associate to HWR(I) and HWR(II) the mode operators $Q^+_{s+n} = Q_{s+n}$ and $Q^-_{-s-n} = \bar{Q}_{s+n}$ and using $SO(1,2)$ tensor product properties, one may build, under some assumptions, an extension $\mathbf{S}$ of the $so(1,2)$ algebra going beyond the standard supersymmetric one. To that purpose, note first that the system J_0, J_+ ,J_- and Q_{s+n} obey the following commutation relations ($s=-1/k$).

$$\begin{aligned}
[J_0, Q_{s+n}] &= (s+n)Q_{s+n} \\
[J_+, Q_{s+n}] &= \sqrt{(2s+n)(n+1)}Q_{s+n+1} \qquad (4.9) \\
[J_-, Q_{s+n}] &= \sqrt{(2s+n-1)n}Q_{s+n-1}
\end{aligned}$$

Similarly we have for the antiholomorphic sector:

$$\begin{aligned}
[\bar{J}_0, \bar{Q}_{s+n}] &= -(s+n)\bar{Q}_{s+n} \\
[\bar{J}_+, \bar{Q}_{s+n}] &= -\sqrt{(2s+n-1)n}\bar{Q}_{s+n-1} \qquad (4.10) \\
[\bar{J}_-, \bar{Q}_{s+n}] &= -\sqrt{(2s+n)(n+1)}\bar{Q}_{s+n+1}
\end{aligned}$$

To close these commutations relations with the Q_s's through a k-th order product one should fullfil the following constraints.
(1) the generalized algebra $\mathbf{S}$ we are looking for should be a generalisation of what is known in two dimensions, i.e a generalisation of 2d fsusy.
(2) When the charge operator Q_{s+n} goes arround an other, say Q_{s+m}, one picks a phase $\Phi = 2i\pi/k$; i.e:

$$Q_{s+n}Q_{s+m} = e^{\pm 2i\pi s}Q_{s+m}Q_{s+n} + \ldots; \quad s = -\frac{1}{k}, \qquad (4.11)$$

where the dots refer for possible extra charge operators of total J_0 eigenvalue (2s+n+m). Eq (4.11) shows also that

the algebra we are searching has a $\mathbf{Z}_k$ graduation. Under this discrete symmetry, Q_{s+n} carries a $+1(modk)$ charge while the $P_{0,\pm}$ energy momentum components have a zero charge mod k.

(3) the generalised algebra $\mathbf{S}$ should split into a bosonic B part and an anyonic A and may be written as: $\mathbf{S} = \oplus_{r=0}^{k-1} A_r = B \oplus_{r=1}^{k-1} A_r$. Since $A_n A_m \subset A_{(n+m)(modk)}$ one has:

$$\begin{aligned} \{A_r \ldots A_r\}_k &\subset B \\ [B, A] &\subset A \\ [B, B] &\subset B. \end{aligned} \tag{4.12}$$

In these eqs, $\{A_r \ldots A_r\}_k$ means the complete symmetrisation of the k anyonic operators A_r and is defined as:

$$\{A_{s_r} \ldots A_{s_r}\}_k = \frac{1}{k!} \sum_{\sigma \in \Sigma} (A_{s_{\sigma(1)}} \ldots A_{s_{\sigma(k)}}) \tag{4.13}$$

where the sum is carried over the k elements of the permutation group $\{1, \ldots, k\}$.

(4) the algebra $\mathbf{S}$ should obey generalised Jacobi identities; in particular one should have:

$$adB\{A_{s_1} \ldots A_{s_k}\} = 0, \tag{4.14}$$

where B stands for the bosonic generators $J_{0,\pm}$ or $P_{0,\pm}$ of the Poincaré algebra. Using eq(4.12) to write $\{A_r \ldots A_r\}_k$ as $\alpha_\mu P^\mu + \beta_\mu J^\mu$ where α and β are real constants; then putting back into the above relation we find that $\{A_r \ldots A_r\}_k$ is proportional to P_μ only. In other words, β_μ should be equal to zero; a property which is easily seen by taking $B = P_\mu$ in eq (4.14). Put differently the symmetric product of the $D_s^\pm$, denoted hereafter as $S^k[D_s^\pm]$, contains the space time vector representation D_1 of $so(1,2)$ and so the primitive charge operators $Q_{-1/k}$ and $\bar{Q}_{1/k}$ obey:

$$\begin{aligned} [J_0, (Q_{-1/k})^k] &= -(Q_{-1/k})^k \sim P_- \\ [J_-, (Q_{-1/k})^k] &= 0 \end{aligned} \tag{4.15}$$

Similarly we have:

$$\begin{aligned} [\bar{J}_0, (\bar{Q}_{1/k})^k] &= (\bar{Q}_{1/k})^k \sim P_+ \\ [\bar{J}_+, (\bar{Q}_{1/k})^k] &= 0 \end{aligned} \tag{4.16}$$

Moreover acting on $(Q_{-1/k})^k$ by adJ_+^n and on $(\bar{Q}_{1/k})^k$ by $ad\bar{J}_+^n$, one obtains:

$$\begin{aligned} adJ_+ (Q_{-1/k})^k &\sim P_0 \\ ad\bar{J}_- (\bar{Q}_{1/k})^k &\sim P_0 \\ ad^2 J_+ (Q_{-1/k})^k &\sim P_- \\ ad^2 \bar{J}_- (\bar{Q}_{1/k})^k &\sim P_+. \end{aligned} \tag{4.17}$$

In summary, starting from $P_{(1,2)}$ and the two Verma modules HWR(I) and HWR(II) (4.3-4), one may build the following new extended symmetry:

$$\begin{aligned} \{Q^\pm_{-\frac{1}{k}}, Q^\pm_{-\frac{1}{k}}, \ldots, Q^\pm_{-\frac{1}{k}}\}_k &= P_\mp = P_1 \pm i P_2 \\ \{Q^\pm_{-\frac{1}{k}}, \ldots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}\}_k &= \pm i \sqrt{\frac{2}{k}} P_0 \\ -(k-1)\{Q^\pm_{-\frac{1}{k}}, \ldots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}\}_k & \\ \pm i\sqrt{k-2}\{Q^\pm_{-\frac{1}{k}}, \ldots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}, Q^\pm_{2-\frac{1}{k}}\}_k &= P_\pm \\ [J^\pm, [J^\pm, [J^\pm, (Q^\pm_{-\frac{1}{k}})^k]]] &= 0 \end{aligned} \tag{4.18}$$

$$\ldots$$

These eqs define the RdTS algebra. For more details on this algebraic structure, see [1,23].

B. More on RdTS invariance

Having described RdTS symmetry, we turn now to make three comments on the algebra (4.18).

1. Bulk HWR

A generic highest weight representation of the RdTS symmetry is obtained, as usual, by successive applications of the creation operators on a given HW state $|\Lambda\rangle$. To work out these representations in a tricky way, note the two following:

(1) eqs(4.18) has two subalgebras $A^+_{(1+2)d}$ and $A^-_{(1+2)d}$ generated by $(J_{0,\pm}, P_{0,\pm}, \mathbf{Q}^+_{-\frac{r}{k}}$, with $0 < r < k$) and $(J_{0,\pm}, P_{0,\pm}, \mathbf{Q}^-_{-\frac{r}{k}}$, with $0 < r < k$) respectively.Thus starting from the vaccum states $|\Lambda_s^\pm\rangle$ in the $(\pm s)$ spin representations of so(1,2), one can build the on shell HW representations of $A^\pm_{(1+2)d}$ which read as:

(i) HWR of $A^+_{(1+2)d}$

$$\begin{aligned} \mathbf{Q}^-_{-\frac{1}{k}} |\Omega_s^+\rangle \ &= \ 0 \\ \{\frac{(\mathbf{Q}^+_{-\frac{1}{k}})^r}{[r]!} |\Omega_s^+\rangle \quad &; \quad o \le r \le k-1\} \end{aligned} \tag{4.19}$$

Note that [r] is the q-number given by $\frac{\omega^{-r} - \omega^r}{\omega^{-1} - \omega^1}$, with $\omega = expi\frac{\pi}{k}$. Note also that the above basis states is k dimensional and carries the spin values $(s - \frac{r}{k})$.

(ii) HWR of $A^-_{(1+2)d}$

$$\begin{aligned} \mathbf{Q}^+_{-\frac{1}{k}} |\Omega_{-s}^-\rangle \ &= \ 0 \\ \{\frac{(\mathbf{Q}^-_{-\frac{1}{k}})^r}{[r]!} |\Omega_{-s}^-\rangle \quad &; \quad o \le r \le k-1\} \end{aligned} \tag{4.20}$$

In this case the basis states carry opposite values of the spin; i.e $(-s + \frac{r}{k})$.

(**2**) A class of HW representations of the full algebra may

be obtained from the above ones just by requiring CPT invariance; in particular by imposing,

$$\begin{aligned} \mathbf{Q}^-_{-\frac{1}{k}} &= [\mathbf{Q}^+_{-\frac{1}{k}}]^+ \\ \mathbf{Q}^-_{1-\frac{1}{k}} &= [\mathbf{Q}^+_{1-\frac{1}{k}}]^+ \end{aligned} \tag{4.21}$$

In the rest frame where $P_0 = E$ is a constant and $J_\pm = P_\pm = 0$, HW representations of $A^+_{(1+2)d}$ have k dimensions and depend on the value of E. For $E = 0$, the representations are nilpotent since $(\mathbf{Q}^+_{-1/k})^k = 0$. However for E non zero, $A^+_{(1+2)d}$ HWRs we are interested in are still nilpotent but have a non trivial center. Examples of $A^+_{(1+2)d}$ representations have been already studied in [1] using finite dimensional matrix representations and differential operators on the space of functions on $R^{1,2}$. A simple example of matrix representation is given by,

$$\begin{aligned} \mathbf{Q}^+_{-\frac{1}{k}} &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \sqrt{[1]} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{[2]} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{[k-1]} & 0 \end{pmatrix}, \\ \mathbf{Q}^+_{1-\frac{1}{k}} &= \begin{pmatrix} 0 & 0 & 0 & \dots & \{\sqrt{[k-1]!}\}^{-1} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.22}$$

2. *Edge HWR*

If one supposes that the RdTS algebra given above is also valid for some class of (1+2) dimensional manifold M with a non zero boundary ∂M, then one may also define the limit of eqs(4.18) on the border of M. To derive this limit and its representations on ∂M, one has to first specify the constraints giving the limit ∂M, identify its right variables; then replace the generators of the bulk symmetry (4.18) by the appropriate variables on ∂M. This operation depends however on the nature of the manifold M if one wants to write down explicit formula. Nevertheless if we admit that on the border of M, we have a two dimensional Poincaré invariance with $SO(2)$ as a Lorentz subgroup; say the little group of $SO(1,2)$ for example, we may obtain a limit of the RdTS algebra on ∂M just by requiring that on the border of M, one is allowed to naively set $P_0 = J_\pm = 0$ in eq(4.18). Roughly speaking the limiting RdTS algebra on ∂M could be then defined as:

$$\begin{aligned} \{\mathbf{q}^\pm_{-\frac{1}{k}}, \dots, \mathbf{q}^\pm_{-\frac{1}{k}}\}_k &= P_\mp \\ [J_0, (\mathbf{q}_{-1/k})^k] &= -(\mathbf{q}_{-1/k})^k \\ [\bar{J}_0, (\bar{\mathbf{q}}_{1/k})^k] &= +(\bar{\mathbf{q}}_{1/k})^k \\ [J_0, P_\pm] &= 0 \\ [\bar{J}_0, P_\pm] &= 0 \\ [\mathbf{q}^\pm_{-1/k}, P_\pm] &= 0 \end{aligned} \tag{4.23}$$

Like in the bulk of M, the above algebra admits two subalgebras A^+_{2d} and A^-_{2d} respectively generated by $(J_0, P_\pm, \mathbf{q}^+_{-\frac{r}{k}}$,with $0 < r < k)$ and $(J_0, P_\pm, \mathbf{q}^-_{-\frac{r}{k}}$,with $0 < r < k)$. HW representations of the algebra (4.23), to which we have refered to as RdTS edge representations, are very special since they usually factor into left and right terms. Moreover EHWS of the RdTS algebra may be obtained from BHWR by going to ∂M and taking the appropriate limits. These representations are quite similar to those used in the study of 2d fsusy [3-7]. A system of vectors basis of such representations may naively written down by considering successive applications by $\mathbf{q}_{-\frac{1}{k}}$ (resp $\bar{\mathbf{q}}_{-\frac{1}{k}}$) on a highest weight state λ. The resulting multiplet of vector basis is then,

$$[(\mathbf{q}^\pm_{-1/k})^r|\lambda\rangle, 0 < r < (k+1)] \tag{4.24}$$

Similar quantities may also be written down for the right sector and then for the full algebra by imposing CPT invariance.

3. *Case where* **M** *is* $\mathbf{AdS_3}$

The consideration of the space time manifold M as AdS_3 is due to the fact that the AdS_3 geometry has many relevant features for the study of RdTS invariance. We propose to review hereafter some of its useful properties:
(i) In the euclidean representation, AdS_3 has an $SO(1,3)$ isometry group containing naturally the $SO(1,2)$ Lorentz symmetry of the (1+2)dimensional $R^{1,2}$ space time.
(ii) AdS_3has a boundary space which may be thought of as the real two-sphere. As one knows, it lives on the two-sphere boundary space time conformal field theories.
(iii) The boundary invariance on ∂AdS_3 has $so(1,2)$ projective subsymmetries which we will relate to the $so(1,2)$ Lorentz subgroup of the RdTS Symmetry.
Using these features we have shown in [8] that the two so(1,2) modules HWR(I) and HWR(II), considered in the building of RdTS supersymmetry, are just special representations of the AdS_3 boundary CFT. To see this relationship, let us review briefly some elements of AdS_3 geometry.

a. $\mathbf{AdS_3}$ manifold

The AdS_3 space is given by the hyperbolic coset manifold $Sl(2,C)/SU(2)$ which may be thought of as the three dimensional hypersurface ${H_3}^+$

$$-{X_0}^2+{X_1}^2+{X_2}^2+{X_3}^2=-l^2 \qquad (4.25)$$

embedded in flat $R^{1,3}$ with local coordinates X^0, X^1, X^2, X^3 . This hypersurface describes a space with a constant negative curvature $(-\frac{1}{l^2})$. The parameter l is choosen to be quantized in terms of the l_s fundamental string lenght units; i.e, $l=l_s\times k$ where k is an integer to be interpreted later on as the Kac Moody level of the $so_k(1,2)$ affine symmetry. To study the field theory on the boundary of AdS_3, it is convenient to introduce the following set of local coordinates of AdS_3:

$$\begin{aligned}
\phi &= log(X_0+X_3)/l \\
\gamma &= \frac{X_2+iX_0}{X_0+iX_3} \\
\bar{\gamma} &= \frac{X_2-iX_1}{X_0+iX_3}
\end{aligned} \qquad (4.26)$$

An equivalent description of the hypersurface is:

$$\begin{aligned}
\gamma &= \frac{r}{\sqrt{l^2+r^2}}e^{-\tau+i\theta} \\
\bar{\gamma} &= \frac{r}{\sqrt{l^2+r^2}}e^{-\tau-i\theta} \\
\phi &= \tau+1/2log(1+r^2/l^2) \\
r &= le^{\phi}\sqrt{\gamma\bar{\gamma}} \\
\tau &= \phi-1/2log(1+e^{2\phi}\gamma\bar{\gamma}) \\
\theta &= \frac{1}{2i}log(\gamma/\bar{\gamma}),
\end{aligned} \qquad (4.27)$$

where we have used the change of variables:

$$\begin{aligned}
X_0 &= X_0(r,\tau)=\sqrt{l^2+r^2}cosh\tau \\
X_3 &= X_3(r,\tau)=\sqrt{l^2+r^2}sinh\tau \\
X_1 &= X_1(r,\theta)=rsin\theta \\
X_2 &= X_2(r,\theta)=rcos\theta
\end{aligned} \qquad (4.28)$$

In the coordinates $(\phi,\gamma,\bar{\gamma})$, the metric of H_3^+ reads as:

$$ds^2=k(d\Phi^2+e^{2\Phi}d\gamma d\bar{\gamma}) \qquad (4.29)$$

Note that in the $(\phi,\gamma,\bar{\gamma})$ frame, the boundary of euclidean AdS_3 corresponds to take the field Φ to infinity. As shown on the above eqs,this is a two sphere which is locally isomorphic to the complex plane parametrized by $(\gamma,\bar{\gamma})$.

b. Strings on $\mathbf{AdS_3}$

Quantum field theory on the AdS_3 space is very special and has very remarkable features governed by the Maldacena correspondence in the zero slope limit of string theory[31]. On this space it has been shown that bulk correlations functions of quantum fields find natural interpretations in the conformal field theory on the boundary of AdS_3[32]. In algebraic language, this correspondance transforms world sheet symmetries of strings on AdS_3 into space time infinite dimensional invariances on the boundary of AdS_3. In what follows we give some relevant results.

b.1 General

To work out explicit field theoretical realisations of these symmetries, we start by recalling that in the presence of the Neveu-Schwarz $B_{\mu\nu}$ field with euclidean world sheet parameterized $(z,\bar{z})$, the dynamics of the bosonic string on AdS_3 is described by the following classical lagrangian:

$$L=k[\partial\Phi\bar{\partial}\Phi+e^{2\Phi}\partial\gamma\partial\bar{\gamma}] \qquad (4.30)$$

In this eq ∂ and $\bar{\partial}$ stand for derivatives with respect to z and $\bar{z}$ repectively. Introducing two auxiliary variables β and $\bar{\beta}$, the above eq may be put into the following convenient form:

$$L^{'}=k^2(\partial\Phi\bar{\partial}\Phi+\beta\bar{\partial}\gamma+\bar{\beta}\partial\bar{\gamma}-e^{-2\Phi}\beta\bar{\beta}) \qquad (4.31)$$

The eqs of motion of the various fields one gets from eq(4.31) read as:

$$\begin{aligned}
\partial\bar{\partial}\Phi-2\beta\bar{\beta}e^{-2\Phi} &= 0 \\
\bar{\partial}\gamma-\beta e^{-2\Phi} &= 0 \\
\partial\bar{\gamma}-\bar{\beta}e^{-2\Phi} &= 0 \\
\partial\bar{\beta}=\bar{\partial}\beta &= 0
\end{aligned} \qquad (4.32)$$

String dynamics on the boundary of AdS_3 is obtained from the previous bulk eqs by taking the limit Φ goes to infinity. This gives:

$$\begin{aligned}
\partial\bar{\partial}\Phi &= 0 \\
\bar{\partial}\gamma=\partial\bar{\gamma} &= 0 \\
\partial\bar{\beta}=\bar{\partial}\beta &= 0
\end{aligned} \qquad (4.33)$$

The world sheet (WS) fields Φ,γ and $\bar{\gamma}$ which had general expressions in the bulk become now holomorphic on the boundary of AdS_3 and describe a boundaryCFT. Note that consistency of quantum mechanics of the string propagating in space time requires that the target space should be $AdS_3\times N$, where N is a (3+n) dimensional compact manifold. To fix the ideas, N may be thought of as $S^3\times T^n$ with $n=20$ for the bosonic string and $n=4$ for superstrings.

b.2 WS Symmetries

World sheet invariances include affine Kac-Moody, Virasoro symmetries and their extensions. For a bosonic string propagating on $AdS_3\times S^3\times T^{20}$, we have the following:

α. Three kinds of WS affine Kac-Moody invariances:

(**i**) A level $(k-2)$ $sl(2)\times s\bar{l(2)}$ invariance coming from the string propagation on AdS_3. This invariance is generated by the conserved currents $J^q_{sl(2)}$ and $\bar{J}^q_{sl(2)}; q=0,\pm1$. In terms of the WS fields $\Phi,\gamma,\bar{\gamma},\beta$ and $\bar{\beta}$ of eq(4.32), the

field theoretical realization of these currents is given by the Wakimoto representation:

$$\begin{aligned} J^-(z) &= \beta(z) \\ J^+(z) &= \beta\gamma^2 + \sqrt{2(k-2)}\gamma\partial\Phi + k\partial\gamma \\ J^0(z) &= \beta\gamma + 1/2\sqrt{2(k-2)}\partial\Phi \\ \bar{J}^-(\bar{z}) &= \bar{\beta} \\ \bar{J}^0(\bar{z}) &= \bar{\beta}\bar{\gamma} + 1/2\sqrt{2(k-2)}\partial\Phi \\ \bar{J}^+(\bar{z}) &= \bar{\beta}\bar{\gamma}^2 + \sqrt{2(k-2)}\bar{\gamma}\partial\Phi + k\partial\bar{\gamma} \end{aligned} \tag{4.34}$$

(**ii**) A level $(k+2)$ invariance coming from the string propagation on S^3. The conserved currents are $J^q_{su(2)}$ and $\bar{J}^q_{su(2)}$. The WS field theoretical realization of these currents is given by the level $(k+2)$ WZW $su(2)$ model [33].

(**iii**) A $u(1)^{20}\times\bar{u}(1)^{20}$ invariance coming from the torus T^{20}. This symmetry is generated by 20 $U(1)$ Kac Moody currents $J^i_{u(1)}; i = 1, \ldots, 20$.

β. WS Virasoro symmetry

This symmetry, which splits into holomorphic and antiholomorphic sectors, is given by the Suggawara construction using quadratic Casimirs of the previous WS affine Kac Moody algebras. For the holomorphic sector, the WS Virasoro currents of a bosonic string on $AdS_3 \times S^3 \times T^{20}$ are:

(**i**) String on AdS_3:

$$T^{WS}_{sl(2)} = \frac{1}{(k-2)}[(J^0_{sl(2)})^2 - (J^1_{sl(2)})^2 - (J^2_{sl(2)})^2] \tag{4.35}$$

(**ii**) String on S^3:

$$T^{WS}_{su(2)} = \frac{1}{(k+2)}[(J^0_{su(2)})^2 + (J^1_{su(2)})^2 + (J^2_{su(2)})^2] \tag{4.36}$$

(**iii**) String on T^{20}:

$$T^{WS}_{u(1)} = \sum_{i=1}^{20}[J^i_{u(1)}]^2 \tag{4.37}$$

Similar quantities are also valid for the antiholomorphic sector of the conformal invariance. Note that the total WS energy momentum tensor T^{WS}_{tot} is given by the sum of $T^{WS}_{sl(2)}, T^{WS}_{su(2)}$ and $T^{WS}_{u(1)}$.

In the case of a superstring propagating on $AdS_3{\times}S^3{\times}T^4$, the above conserved currents are slightly modified by the adjunction of extra terms due to contributions of WS fermions .

V. INFINITE SPACE-TIME INVARIANCES OF ADS$_3$

To analyse the space-time infinite dimensional symmetries on the boundary of AdS_3, one may follow the same strategy that we have used for the study of WS invariances. First identify the space time affine Kac-Moody symmetries and then consider the space time conformal invariance and eventually the Casimirs of higher ranks. We shall simplify a little bit the analysis of space-time invariance and focus our attention on the conformal symmetry on $\partial(AdS_3)$. Some specific features on space time Kac-Moody symmetries will also be given in due time.

We begin by noting that space time infinite invariances on the boundary of AdS_3 are intimately linked to the WS ones. For the case of a bosonic string propagating on $AdS_3{\times}S^3{\times}T^{20}$, we have already shown that there are various kinds of WS symmetries coming from the propagation on AdS_3, S^3 and T^{20} respectively. In the ϕ infinite limit, we can show that one may use these WS symmetries to build new space time ones.

A. Space time conformal invariance

First of all, note that the global part of the WS $SO(1,2){\times}SO(1,2)$ affine invariance of a bosonic string on AdS_3, generated by J^q_0 and $\bar{J}^q_0; q = 0, \pm 1$ may be realized in different ways. A tricky way, which turns out to be crucial in building space-time conformal invariance, is given by the Wakimoto realization [34]. Classically, this representation reads in terms of the local coordinates $(\Phi, \gamma, \bar{\gamma})$ as follows:

$$\begin{aligned} J^0_0 &= \gamma\partial/\partial\gamma - 1/2\partial/\partial\gamma, \\ J^-_0 &= \partial/\partial\gamma, \\ J^+_0 &= \gamma^2\partial/\partial\gamma - \gamma\partial/\partial\Phi - e^{-2\Phi}\partial/\partial\gamma. \end{aligned} \tag{5.1}$$

Similar relations are also valid for $\bar{J}^q_0$; they are obtained by substituting γ by $\bar{\gamma}$. Quantum mechanically, the charge operators J^q_0 and $\bar{J}^q_0$ are given in terms of the Laurent mode operators of the quantum fields $\Phi, \gamma, \bar{\gamma}, \beta$ and $\bar{\beta}$ by using eqs(4.36) and performing the Cauchy integrations:

$$\begin{aligned} J^q_0 &= \int \tfrac{dz}{2i\pi} J^q(z) \\ \bar{J}^q_0 &= \int \tfrac{d\bar{z}}{2i\pi} \bar{J}^q(\bar{z}). \end{aligned} \tag{5.2}$$

To build the space time conformal invariance on the AdS_3 boundary, we proceed by steps. First suppose that there exists really a conformal symmetry on the boundary of AdS_3 and denote the space time Virasoro generators by L_n and $\bar{L}_n, n{\in}Z$. The L_n and $\bar{L}_n$, which should not be confused with the WS conformal mode generators, satisfy obviously the left and right Virasoro algebras.

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \tfrac{c}{12}n(n^2-1)\delta_{n+m} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \tfrac{c}{12}n(n^2-1)\delta_{n+m} \\ [L_n, \bar{L}_m] &= 0. \end{aligned} \tag{5.3}$$

The second step is to solve these eqs by using the string WS fields $(\Phi, \gamma, \bar{\gamma})$ on AdS_3. To do so, it is convenient to divide the above eqs into two blocks. The first block

corresponds to set $n = 0, \pm 1$ in the generators L_n and $\bar{L}_n$ of eqs(6.3). It describes the anomaly free projective subsymmetry the Virasoro algebra. The second block concerns the generators associated with the remaining values of n.
On the boundary of AdS_3 obtained by taking the infinite limit of the Φ field, one solves the projective subsymmetry by natural identification of L_q and $\bar{L}_q$; $q = 0, \pm 1$ with the zero modes of the WS $so(1,2) \times \bar{so}(1,2)$ affine invariance. In other words we have:

$$\begin{array}{lll} L_q & = & -\int \frac{dz}{2i\pi} J^q(z) = -J_0^q; \quad q = 0, \pm 1 \\ \bar{L}_q & = & \int \frac{d\bar{z}}{2i\pi} \bar{J}^q(\bar{z}) = -\bar{J}_0^q; \quad q = 0, \pm 1. \end{array} \tag{5.4}$$

Note that on the AdS_3 boundary, viewed as a complex plane parametrized by $(\gamma, \bar{\gamma})$, the charge operators J_0^- (L_{-1}) and $\bar{J}_0^-(\bar{L}_{-1})$ taken in the Wakimoto representation coincide respectively with the translation operators P_- and $\bar{P}_+$:

$$\begin{array}{l} P_- = L_- = \partial/\partial\gamma \\ P_+ = \bar{L}_- = \partial/\partial\bar{\gamma}. \end{array} \tag{5.5}$$

Eqs(5.4-5) are interesting; they establish a link between the L_- and $\bar{L}_-$ constants of motion of the boundary conformal field theory on AdS_3 on one hand and the P_- and the P_+ translation generators of the RdTS extension of the $so(1,2)$ algebra on the other hand.
To get the rigourous solution of the remaining Virasoro charge operators L_n and $\bar{L}_n$, one has to work hard. This is a lengthy and technical calculation which has been done in [35] in connection with the study of the D_1/D_5 brane system. We shall use an economic path to work out the solution. This is a less rigourous but tricky way to get the same result. This method is based on trying to extend the L_n and $\bar{L}_n$; $n = 0, \pm 1$ projective solution to arbitrary integers n using properties of the string WS fields near the boundary, dimensional arguments and similarities with the photon vertex operator in three dimensions. Indeed using the holomorphic property of γ and $\bar{\gamma}$ eqs(4.33) as well as the space time dimensional arguments;

$$\begin{array}{ll} [\gamma] = -1; & J^0_{sl(2)} = 0 \\ J^-_{sl(2)} = 1; & J^+_{sl(2)} = -1, \end{array} \tag{5.6}$$

it is not difficult to check that the following $L_n(\bar{L}_n)$ expressions are good condidates:

$$\mathbf{L}_n = \int \frac{dz}{2i\pi}[a_0 \gamma^n J^0_{sl(2)} - \frac{a_-}{2}\gamma^{n+1} J^-_{sl(2)} + \frac{a_+}{2}\gamma^{n-1} J^+_{sl(2)}], \tag{5.7}$$

and a similar relation for $\bar{L}_n$. To get the a_i coefficients, one needs to impose constraints which may be obtained by using results of BRST analysis in QED_3. Following [32],the right constraints one has to impose on the a_i' s are:

$$\begin{array}{l} na_0 + (n+1)a_- + (n-1)a_+ = 0 \\ J^0\gamma - (1/2)J^-\gamma^2 - (1/2)J^+ = 0. \end{array} \tag{5.8}$$

The solution of the first constraint of these eqs reproducing the projective generators (5.4) is as follows:

$$\begin{array}{l} a_0 = (n^2 - 1) \\ a_- = n(n-1) \\ a_+ = n(n+1) \end{array} \tag{5.9}$$

Moreover using the second constraint of eqs(5.8) to express $J^+_{sl(2)}(z)$ in terms of $J^0_{sl(2)}(z)$ and $J^-_{sl(2)}(z)$; then putting back into eqs(5.7), we find:

$$\mathbf{L}_n = \int \frac{dz}{2i\pi}[-(n+1)\gamma^n J^0_{sl(2)} + n\gamma^{n+1} J^-_{sl(2)}]. \tag{5.10}$$

Eqs (5.4) and (5.10) define the space time Virasoro algebra on the boundary of AdS_3.

B. Other symmetries of AdS$_3$

Having built the L_n's space time Virasoro generators, one may be interested in determining the space-time energy momentum tensors $T(\gamma)$ and $\bar{T}(\bar{\gamma})$ of the boundary CFT on AdS_3. It turns out that the right form of the space-time energy momentum tensor depends moreover on auxiliary complex variables $(y, \bar{y})$ so that the space time energy momentum tensor has now two arguments; i.e: $T = T(y, \gamma)$ and $\bar{T} = \bar{T}(\bar{y}, \bar{\gamma})$. Following [33], $T(y, \gamma)$ and $\bar{T}(\bar{y}, \bar{\gamma})$ read as:

$$\begin{array}{l} T(y,\gamma) = \int \frac{dz}{2i\pi}[\frac{\partial_y J(y,\gamma)}{(y-\gamma)^2} - \frac{\partial^2{}_y J(y,\gamma)}{(y-\gamma)}] \\ \bar{T}(\bar{y},\bar{\gamma}) = \int \frac{d\bar{z}}{2i\pi}[\frac{\partial_{\bar{y}} J(\bar{y},\bar{\gamma})}{(\bar{y}-\bar{\gamma})^2} - \frac{\partial^2{}_{\bar{y}} J(\bar{y},\bar{\gamma})}{(\bar{y}-\bar{\gamma})}], \end{array} \tag{5.11}$$

where the currents $J(y, \gamma)$ and $J(\bar{y}, \bar{\gamma})$ are given by:

$$J(y,\gamma) = -J^+(y,\gamma) = 2yJ^0(\gamma) - J^+(\gamma) - y^2 J^-(\gamma). \tag{5.12}$$

In connection to these eqs, it is interesting to note that the conserved currents $J^q(y, \gamma)$ and $J^q(\bar{y}, \bar{\gamma})$ are related to the WS affine Kac-Moody ones on AdS_3 as follows:

$$\begin{array}{lll} J^+(y,\gamma) & = & e^{-yJ_0^-} J^+(\gamma) e^{yJ_0^-} \\ & = & J^+(\gamma) - 2yJ^0(\gamma) + y^2 J^-(\gamma) \\ J^0(y,\gamma) & = & e^{-yJ_0^-} J^0(\gamma) e^{yJ_0^-} \\ & = & J^0(\gamma) - yJ^-(\gamma) = -\frac{1}{2}\partial_z J^+(y,\gamma) \\ J^-(y,\gamma) & = & e^{-yJ_0^-} J^-(\gamma) e^{yJ_0^-} \\ & = & J^-(\gamma) = \frac{1}{2}\partial_z^2 J^+(y,\gamma) \end{array} \tag{5.13}$$

and analogous eqs for $J^q(\bar{y}, \bar{\gamma})$. Putting eqs(5.12) back into eqs(5.11) and expanding in power series of $\frac{\gamma}{y}$, one discovers the L_n space time Virasoro generators given by eqs(5.10).
Note moreover that one may also build space time affine

Kac-Moody symmeties out of the WS ones. Starting from WS conserved currents $E^a_{ws}(z)$, which may be thought of as $J^q_{sl(2)}(z)$, and going to the boundary of AdS_3, the corresponding space time affine Kac-Moody currents $E^a_{spacetime}(y,\gamma)$ read as:

$$E^a_{spacetime}(y,\gamma) = \oint \frac{dz}{2i\pi}[\frac{E^a_{ws}(z)}{(y-\gamma(z))}]. \tag{5.14}$$

Expanding this eq in powers of $\frac{y}{\gamma}$ or $\frac{\gamma}{y}$, one gets the space time affine Kac-Moody modes:

$$E^{a,spacetime}_n = \oint \frac{dz}{2i\pi}[E^a_{ws}(z)\gamma^n]. \tag{5.15}$$

Note finally that RdTS symmetry may be thought of as just integral deformations of WS and space time symmeties on AdS_3. Such procedure is standard in the study of integrable models obtained from deformations of CFTs by relevant perturbations. To see how this works in our case let us give some details. If one forgets about string dynamics as well as the nature of the compact manifold N and just retains that on $\partial(AdS_3)$ lives a conformal structure, one may consider its highest weight representations which read as in eqs(3.23). A priori the central charge c and the conformal weights h and $\bar{h}$ of these representations are arbitrary. However requiring unitary conditions, the parameters c, h and $\bar{h}$ are subject to constraints which become more stronger if one imposes extra symmetries such as supersymmetry or parafermionic invariance [36]. Having these details in mind, one may also build descendant states $|h+n,\bar{h}+\bar{n}\rangle$ of $|h,\bar{h}\rangle$ from the primary ones as follows,

$$|h+n,\bar{h}+\bar{n}\rangle = \sum_{\substack{n=\sum\alpha_i n_i\\ \bar{n}=\sum\beta_j n_j}} \lambda_{\{\alpha_i\}\{\beta_j\}}(\Pi_i L^{\alpha i}_{-n_i})(\Pi_j \bar{L}^{\beta j}_{-n_j})|h,\bar{h}\rangle. \tag{5.16}$$

where the α_i's and β_j's are positive integers and $\lambda_{\alpha\beta}$ are C-numbers which we use to denote the collective coefficients $\lambda_{\{\alpha_i\}\{\beta_j\}}$. They satisfy the following obvious relations.

$$\begin{array}{l} L_0|h+n,\bar{h}+\bar{n}\rangle = (h+n)|h+n,\bar{h}+\bar{n}\rangle \\ L_\pm|h+n,\bar{h}+\bar{n}\rangle = a_\pm(h,n)|h\pm n,\bar{h}\pm\bar{n}\rangle \\ \bar{L}_0|h+n,\bar{h}+\bar{n}\rangle = (\bar{h}+\bar{n})|h+n,\bar{h}+\bar{n}\rangle \\ \bar{L}_\pm|h+n,\bar{h}+\bar{n}\rangle = \bar{a}_\pm(\bar{h},\bar{n})|h\pm n,\bar{h}\pm\bar{n}\rangle, \end{array} \tag{5.17}$$

where $a_\pm(h,n)$ and $\bar{a}_\pm(\bar{h},\bar{n})$ are normalization factors. Making an appropriate choice of the $\lambda_{\alpha\beta}$ coefficients and taking the $a_\pm(h,n)$ and $\bar{a}_\pm(h,\bar{n})$ coefficients as given herebelow,

$$\begin{array}{l} a_-(h,n) = \sqrt{(2h+n)(n+1)} \\ a_+(h,n) = \sqrt{(2h+n-1)n}, \end{array} \tag{5.18}$$

one gets the two $so(1,2)$ modules eqs(4.9-10) used in building RdTS supersymmetry. As a summary we should retain :
(α). the RdTS extension of Poincaré invariance in (1+2) dimensions we have been describing is a special kind of fsusy algebra. It is a residual symmetry of a boundary space time conformal invariance living on ∂M. Here $M = AdS_3$.
(β). the explicit analysis of this paper has been made possible due to the particular properties of the AdS_3 geometry since,
(**i**)- the AdS_3 manifold carries naturally a $so(1,2)$ affine invariance which has various realisation ways.
(**ii**)- the Wakimoto realisation of the $SO(1,2)$ affine symmetry which on one hand relates its zero mode to the projective symmetry of a boundary CFT on AdS_3 and on the other hand links the L_- and $\bar{L}_-$ to the translation operators on $\partial(AdS_3)$ as shown on eqs (5.4).
(**iii**)- the correspondance between WS and space time symmetries which plays a crucial role in analysing the various kinds of symmetries living on $\partial(AdS_3)$.
In the end we want to that the comodity in using AdS_3 geometry in the above algebraic approach should be compared to the droplet approximation of the CS effective field theory of FQH liquids.

VI. FQHS/RDTS CORRESPONDANCE

Let us start by recalling the different types of exotic quantum states that we have encountered in the study FQHE systems with boundaries and in the analysis of RdTS representations.

A. FQH states

In the CS effective model of FQHE, we have distinguished two kinds of quantum states carrying fractional values of the spin and the electric charge. These are the Bulk states and edge ones.
(**1**) Bulk states are localised states carrying fractional spins $\frac{\theta}{\pi}$ as in eqs (2.13) and (3.6) and playing a crucial role in (1+2)dimensional effective CS abelian $U(1)^k$ gauge model, especially in the study of hierarchies. Classically these states might be thought of as associated with Wilson lines of the CS gauge fields. Quantum mechanically, these states should be viewed as representation states of q-deformed algebras of creation and annihilation operators of certain quantum fields. However a such quantum fields formulation is still far from reach as no consistent local quantum field model has been built yet. Tentatives towards developping a q-quantum field operators generating quantum states carrying fractional values of the spin have been considered in different occasions in the past; for a review on some field and algebraic methods, see [37]. For our concern, we shall make, In subsection 6.3, a hypothesis regarding this matter by suggesting the RdTS symmetry as the algebra of these quantum

states.
(**2**) Edge states are extended states carrying also fractional values of the spin and are nicely described, in the droplet approximation, by a boundary conformal field theory. Edge states are quantum states playing an important role first because they are responsible for the quantization of the Hall conductivity σ_{xy} and second for their non trivial dynamics on the boundary.

B. RdTS states

In the language of RdTS representations, we have also distinguished bulk representations living on M and edge representations living on its border ∂M.
(**1**) Bulk states are highest states of the HWRs of the algebra (4.18). They are given by CPT invariant representations of the $A^{\pm}_{(1+2)d}$,:

$$\begin{array}{l} \frac{(\mathbf{Q^{+}}_{-\frac{1}{k}})^{\mathbf{r}}}{[\mathbf{r}]!}|\Omega_s^{+}\rangle, \frac{(\mathbf{Q^{-}}_{-\frac{1}{k}})^{\mathbf{r}}}{[\mathbf{r}]!}|\Omega^{-}_{-s}\rangle, \\ \mathbf{Q^{\mp}}_{-\frac{1}{k}}|\mathbf{\Omega_s^{\pm}}\rangle = \mathbf{0}, \mathbf{Q^{-}}_{-\frac{1}{k}} = [\mathbf{Q^{+}}_{-\frac{1}{k}}]^{+} \\ \mathbf{Q^{-}}_{\mathbf{1}-\frac{1}{k}} = [\mathbf{Q^{+}}_{\mathbf{1}-\frac{1}{k}}]^{+}, \end{array} \tag{6.1}$$

where $o \leq r \leq k-1$.
(**2**) Edge states are highest states of the HWRs of the algebra (4.23).They read as follows,

$$[(\mathbf{q}^{\pm}_{-1/k})^{r}|\lambda\rangle, 0 < r < (k+1)]. \tag{6.2}$$

C. Correspondence

We have learned in subsection 2.2 that in the droplet approximation, egde excitations of FQH droplets are described by conformal vertex operators carrying fractional values of the spin. These vertices live on the boundary of the disk geometry of the droplet. Abstraction done from the value of the conformal spin, the droplet vertices look like the tachyon vertex of string field theory; their correlations and scatterings may be then studied by using similar methods as those developed in the context of open string field theory [38].
Outside the droplet approximation where the boundary conformal invariance is not exact, one expects that some constants of motion carrying fractional values of the spin are still conserved and are the generators of the exotic quantum excitations one encounters in the effective CS gauge model of FQHE. If one accepts this reasonning, it follows then that there is a natural correpondence between the edge excitations of FQHE and the RdTS representation states living on ∂M. This conjecture is also supported by the fact that representations on ∂M for both FQH liquids in the droplet approximation and RdTS symmetry on AdS_3 are associated with boundary conformal invariances on ∂M. Beyond these approximations, the boundary conformal invariance is no longuer exact and one is left with residual symmetries and constants of motion carrying fractional values of the spin. In other words we expect that one may have the following natural correspondence:
(1) bulk states involved in the CS effective gauge model FQHE are associated with bulk HWR of the RdTS algebra. Put differently the algebra (4.18) could be conjectured as a condidate for the algebra of creation and annhilation of quasiparticle states.
(2) Edge states of FQH systems are associated with edge HWR of the limit of RdTS algebra on the border ∂M of the (1+2) dimensional manifold M.
(3) droplet approximation of FQH liquids is associated with the AdS_3 geometry. In other words the edge CFT of FQH droplet is associated with space time CFT on the $\partial(AdS_3)$.

VII. CONCLUSION

In this paper we have mainly studied two things:
First, we have considered the Chern-Simons effective gauge model of FQHE and completed partial results on topological orders of FQH hierachical states. We have shown that, upon performing special $Gl(n,Z)$ transformations on the CS gauge fields, one may build an equivalent effective model having the same filling fraction ν and extending known results on Haldane hierarchy. More specifically, we have shown that any k^{th} level Haldane state of filling fraction ν_H can be usually interpreted as a bound state of k Laughlin states of filling fractions (ν_j); $j = 1,\ldots,k$, where $\nu_j = \frac{1}{m_j m_{j+1}}$ and where the m_j integers are solutions of the series $m_j = p_j m_{j-1} - m_{j-2}$; with $m_0 = 1$ and $m_1 = p_1$. One of the remarkable properties of this decomposition is that the $m_j m_{j+1}$ product is usually an odd integer and so each factor ν_j describes indeed a Laughlin state.
Second we have reviewed the main lines of the RdTS algebra on a (1+2)dimensional manifold M with a boundary ∂M and studied its highest weight representations. We have shown that RdTS algebra has generally two kinds of HWRs: bulk highest weight representations and edge highest weight ones. They live respectively in M and on ∂M. Then using the features of the FQH quasiparticles we encounter in the CS effective field model of FQHE, we have conjectured that they are appropriate condidates to be described by the HWRs of the RdTS algebra. We have given several arguments supporting our hypothesis either by using algebraic correspondence between FQH quasiparticles and RdTS HW states or by working out geometric similarities between the droplet approximation and AdS_3 geometry. In the end we would like to note that CS effective gauge model of FQHE and the RdTS algebra have quite similar boundary invariances. In droplet approximation of the FQH liquids as well as for RdTS problem on the AdS_3 space-time geometry, the above invariances share some general features with the Maldacena

correspondence of strings propagating on Anti-de Sitter backgrounds[3]. We hope to be able to develop this issue in a future occasion and find ways for exploiting results, obtained in the context of strings on $AdS_r \times N^{10-r}$ in particular in the study of the D_1/D_5 system of superstrings on AdS_3 times a seven dimensional compact manifold N^7 [39], in order to analyse further the correspondance we have proposed in section 5.

Acknowledgements

One of us (I.B) would like to thank Prof J Wess for kind hospitality at section der physik theorie, Munich University and CNCPRST (Morocco) and DFG (Germany) for supporting her stay in Munich under the contrat 445 Mar 113/5/0 Germany. This research work has been also supported by the program under PARS Phys 27/ 372/98.

VIII. REFERENCES

[1]M.Rausch de Traubenberg and M.J. Slupinski, Mod.Phys.Lett. A12 (1997) 3051-3066.

[2]M.Rausch de Traubenberg, M. J. Slupinski Fractional Supersymmetry and Fth-Roots of Representations,J.Math.Phys.41(2000)4556-4571. M.Rausch de Traubenberg, Fractional supersymmetry and Lie Algebras, Lectures given at the workshop on string theory and non commutative geometry. To appear in the proceeding of the workshop June 16-17, 2000, Rabat.

[3]A.Leclair, C.Vafa Nucl.Phys. B401(1993)413. D.Bernard, A Leclair, Nucl.Phys. B340(1990)712; Phys.Lett B247 (1991)309; Commun.Math.Phys. 142 99.

[4]E.H.Saidi, M.B.Sedra and J.Zerouaoui, Class.Quant.Grav. 12(1995)1567-1580.

[5]E.H.Saidi, M.B.Sedra and J.Zerouaoui, Class Quant Grav. **12** 2705. A.Kadiri, E.H Saidi, M.B Sedra and J.Zerouaoui 1994 On exotic supersymmetries of the thermal deformation of minimal models ICTP preprintIC/94/216. Perez A, Rausch de Traubenberg M and Simon P 1996 Nucl.Phys.B **482** 325

[6]A.ElFallah,E.H Saidi,and J.Zerouaoui Phys.Lett.B 468(1999)86-95.Chakir,A.ElFallah and E.H Saidi,Mod Phys lett **38** 2931 Class Quant.Grav.**14**(1997)20-40. A. ElFallah, E.H Saidi and R Dick Class Quant.Grav.**17**(2000)43

[7]I.Benkaddour and E.H. Saidi Class.Quantum. Grav.16 (1999)1793 – 1804.

[8]I Benkaddour, A ElRhalami and E.H Saidi "Non Trivial Extension of the (1+2)-Poincaré Algebra and Conformal Invariance on the Boundary of $\mathbf{AdS_3}$" hep-th/0007142

[9]M.D. Johnson and G.S Canright Phys. Rev.B49 (1994)2947

[10]R.E.Prange and S.M. Girvin, The Quantum Hall effect(Springer, New York, 1987), R.B.Laughlin, Phys.rev.Lett. 50(1983)1395.

[11]Xiao-Gang Wen "Topological orders and Edge Escitations in FQH State"; cond-mat/9506066

[12]C.L.Kane and Matthew P.A. Fisher "Impurity scattering and transport of Fractional Quantum Hall Edge States" new v.(july 20.2000);cond-mat/9409028

[13]R.E.Prange " The Quantum Hall Effect" 2d.Ed 1990 chapter (I) J.E.Moore and F.D. Haldane "Edge excitations of the $\nu = \frac{2}{3}$ spin-singlet QHS" cond-mat/9606156

[14]Steven M. Girvin,(Indiana University), "The Quantum Hall Effect: Novel Excitations and Broken Symmetries",cond-mat/9907002.

[15]F.D.M.Haldane, Phys.rev.lett.51,(1983)605

[16]B.I.Halperin, Phys.Rev.Lett.52, (1984)1583

[17]S.Girvin, Phys.Rev.B29,(1984)6012

[18]A.H.Macdonald, D.B.Murray, Phys.Rev.B32 (1985)2707

[19]J.K.Jain,Phys.RevB41(1991)7653

[20]X.G.Wen and Zee, Phys.RevB44 (1991)274

[21]B.Blok and X.G.Wen, Phys.Rev.B42 (1990)8133

[22]J.Frohlich and A.Zee,Nucl.Phys.B364, (1991)517

[23]Eduardo Fradkin, Lecture Note Series 82 (Field Theories of Condensed matter systems) 1991

[24]Shou Cheng Zhang , Int.J.Mod.Phys.B6 (1992)

[25]X.G.Wen,Int.J.Mod.Phys.B2,(1990)

[26]X.G.Wen,Q.Niu,Phys.Rev.B41, (1990)9377

[27]A ElRhalami and E.H Saidi,in preparation

[28]X.G. Wen Dynamics of the Edge excitations in the FQH Effects, In the Proceeding of the Fourth Trieste conference on Quantum Field Theory and Condensed matter Physics; edited by Randjbar-Daemi and Yu.Lu, World scientific (1991).

[29]D.H.Lee and X.G.Wen, Phys.Rev.lett.66, (1991) 1765

[30]J Wess and B. Zumino Nucl Phys B(Proc Suppl.)18(1990)302.
C De Concini and V Kac Prog.Math 92.(1990)471
Daniel Arnaudon, Vladimir Rittenberg, Quantum Chains with $U_q(SL(2))$ Symmetry and Unrestricted Representations, Phys.Lett. B306 (1993) 86-90

[31]J.Maldacena,Adv.Theor.Math.Phys.2 (1997)231.hep-th/9711200.

[32]A.Giveon,D.Kutasov and N.Seiberg, Comments on String Theory on AdS_3, hep-th/9806194

[33]J.Balog,L.O'Raifeartaigh,P. Forgacs,A.Wipf, Nucl.Phys.B325 (1989)225

[34]M.Wakimoto, Comm.Math.Phys.104(1986)605

[35]D.Kutasov, N.Seiberg, More Comments on String Theory on AdS_3,hep-th/9903219, JHEP 9904 (1999) 008

[36]Zamolodchikov AB and Fateev V A 1985 Sov.Phys. -JETP **62** 215
D.Kastor, E.Martinec and Z.Qui, Phys.Lett.B200(1988)134
D.Gepner, Nucl.Phys.B 290(1987),10
Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Nucl. Phys.Lett.B **214** 333

[37]F Wilczek, Fractional statistics and anyon superconductivity, World scientific, singapore (1990)
S Forte, Mod.Phys Lett A6,(1991)3153, Mod Phys 64 (1992) 193.
J.Douari Anyons; statistiques fractionnaires et groupes quantiques, PhD Thesis, Faculty of Sciences, Rabat university (2000).

[38]B.Zweibach, Lecture Notes on string field theory delivered at the spring school on string theory (2000) ICTP Trieste Italy
A.Sen and B.Zweibach hep-th/9912249 JHEP0003 (2000)002.

[39]A Giveon, M.Rocek hep-th/9904024 JHEP 9904(1999)019
R.Argurio, A.Giveon, Assaf Shomer hep-th/0011046, The spectrum of N=3 String theory on $\mathrm{AdS}_3 \times G/H$